Rayleigh Lecture

Metamaterials in Acoustics and Vibration: From Theory to Practice

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concepts

geometry = material properties

via homogenization or transformation or both

Applications:

- **narrow band**: phononic crystals, negative index materials
- **broadband**: acoustic cloaking
overview

Introduction: metamaterials

Phononic crystals – engineering the band gap waves in periodic systems
examples of phononic crystals

Acoustic cloaking – engineering the impossible transformation acoustics
1D, 2D, cylindrical inertial materials
Pentamode materials
cloaking elastic waves

Metatheory for metamaterials?
introduction
Word first used in 1999 by Rodger Walser of the University of Texas, Austin.

Metamaterials are:

*Macroscopic composites having a manmade, three-dimensional, periodic cellular architecture designed to produce an optimized combination, not available in nature, of two or more responses to specific excitation.*
Negative Refraction Makes a Perfect Lens

(Pendry, 2000)

Microwave flat lens

Parimi, Lu, Vodo and Sridhar
Nature 2003

normal behavior
A cloak makes waves travel around an object

Pendry, Schurig and Smith
Science 2006

normal behavior
Metamaterial Electromagnetic Cloak at Microwave Frequencies

Simulation

perfect cloak
imperfect cloak

copper cylinder
cloaked cylinder

Experiment
Why should engineers be interested in *metamaterials*?

potential for radically new devices/technologies

- vibration insulation
- ultrasonic imaging beyond the diffraction limit
- control of SAWs: filtering, guiding, ...
- stealth for underwater structures

**why now?**

combination of:

- cross-disciplinarity
- advanced fabrication techniques
- computational availability
- thinking “outside the box”
Introduction: metamaterials

**Phononic crystals – engineering the band gap**
waves in periodic systems
examples of phononic crystals

Acoustic cloaking – engineering the impossible
transformation acoustics
1D, 2D, cylindrical
inertial materials
Pentamode materials
cloaking elastic waves

Metatheory for metamaterials?
phononic crystals

engineering the band gap
phononic crystals
periodic mechanical systems

phononic band gaps provide tools for controlling waves: filtering, steering, trapping, focusing beyond the diffraction limit, etc.
acoustics

density \( \rho \)

bulk modulus \( K \)

pressure relation \( \dot{p} = -K \text{ div } \mathbf{v} \)

momentum balance \( \rho \dot{\mathbf{v}} = -\nabla p \)

assume constant \( \frac{\rho}{K} \)

\[ \nabla^2 p - \frac{1}{c^2} \ddot{p} = 0 \]

wave speed \( c = \sqrt{\frac{K}{\rho}} \)

water 1500 m/s
air 330 m/s

ratio of pressure to particle velocity in a plane wave \( \frac{p}{v} = \rho c \equiv \text{ acoustic impedance} \)
scattering of sound

planewave

\[ p \sim e^{i(\vec{k} \cdot \vec{x} - \omega t)} \]

\[ |\vec{k}| = \frac{\omega}{c} = \frac{2\pi}{\lambda} \]
...but for some $\lambda$ (~ $2a$), no sound propagates: a phononic band gap

for most $\lambda$, sound propagates through crystal without scattering (scattering cancels coherently)
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Metatheory for metamaterials?
waves in periodic systems

G. Floquet (1883)        F. Bloch (1928)

Bloch-Floquet theorem applies:

\[ u(\vec{x}, t) = e^{i(\vec{k} \cdot \vec{x} - \omega t)} U_{\vec{k}}(\vec{x}) \]

- plane wave
- periodic envelope

\( \vec{k} \) is constant, i.e. no scattering of Bloch wave

\( U_{\vec{k}} \) given by the finite unit cell,
so the frequencies are discrete \( \omega_n(\vec{k}) \)
waves in periodic systems: 1D

$k$ is periodic:

$k + 2\pi/a$ equivalent to $k$

\[ \rho(x) = \rho(x+a) \]

irreducible Brillouin zone
Any 1D periodic system has a gap

first, treat as artificially periodic

bands are “folded” by $2\pi/a$ equivalence
Any 1D periodic system has a gap

still treat it as "artificially" periodic

forward, backward propagating waves
or standing waves

\[ u = u_0 \cos(kx - \omega t) + u_0 \cos(kx + \omega t) = 2u_0 \cos(kx) \cos(\omega t) \]
Any 1d periodic system has a gap

add a small “real” density periodicity

\[ \rho_2 = \rho_1 + \Delta \rho \]

splitting of degeneracy: wave mainly in slower zone \((\rho_2)\) has lower frequency
Rayleigh predicted band gaps in 1887

The meaning of this is that a wave travelling in either direction is ultimately totally reflected. For example, we may so choose the values of $R$ and $S$ that at the origin of $x$ there is a wave (of given strength) in the positive direction only, and we may imagine that it here passes into a uniform medium, and so is propagated on indefinitely without change. But, in order to maintain this state of things, we have to suppose on the negative side the coexistence of positive and negative waves, which at sufficient distances from the origin are of nearly equal and ever-increasing amplitudes. In order therefore that a small wave may emerge at $x = 0$, we have to cause intense waves to be incident upon a face of the medium corresponding to a large negative $x$, of which nearly the whole are reflected.

Rayleigh 1870

ON THE MAINTENANCE OF VIBRATIONS BY FORCES’ OF DOUBLE FREQUENCY, AND ON THE PROPAGATION OF WAVES THROUGH A MEDIUM ENDOWED WITH A PERIODIC STRUCTURE.

[Philosophical Magazine, xxiv. pp. 145—159; 1887.]
Bloch waves – 1D periodic system

time harmonic \[ e^{-i\omega t} \]

\[
\begin{pmatrix}
  \nu' \\
  p'
\end{pmatrix}
= i\omega
\begin{pmatrix}
  0 & K^{-1} \\
  \rho & 0
\end{pmatrix}
\begin{pmatrix}
  \nu \\
  p
\end{pmatrix}
= \frac{i\omega}{c}
\begin{pmatrix}
  0 & z^{-1} \\
  z & 0
\end{pmatrix}
\begin{pmatrix}
  \nu \\
  p
\end{pmatrix}
\]

propagator \[ e^{ikxA} = I \cos kx + iA \sin kx \]
\[ \det e^{ikxA} = 1 \]

two phase medium

Bloch condition \[ \eta(a) = e^{ika} \eta(0) \]

\[
\begin{pmatrix}
  e^{ik_1 a_1 A_1} e^{ik_2 a_2 A_2} - e^{ika} I
\end{pmatrix} \eta(0) = 0
\]

\[ e^{ika} + e^{-ika} = \text{tr} \left( e^{ik_1 a_1 A_1} e^{ik_2 a_2 A_2} \right) \]

dispersion relation \[ \cos ka = \cos k_1 a_1 \cos k_2 a_2 - \sin k_1 a_1 \sin k_2 a_2 \frac{1}{2} \left( \frac{z_1}{z_2} + \frac{z_2}{z_1} \right) \]
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Metathtory for metamaterials?
phononic crystals

example applications
filtering and controlling waves

perfect mirror (in band gap)
- frequency filter

wave guide
- beam splitting, multiplexing

Khelif et al (APL 2004)
phononic crystals for surface acoustic waves

Mohammadi et al. (APL 2008)

cubic arrays give larger gaps than triangular but are not as isotropic
stronger scattering leads to greater band gaps

**Partial band gap** for waves in some directions only

**Complete band gap** - all directions
resonant oscillator metamaterials

mass-in mass system

equivalent system

Compare equilibrium eqns:

\[
\frac{m_{\text{eff}}}{m_{\text{st}}} = 1 + \left( \frac{\theta}{1+\theta} \right) \frac{(\omega/\omega_0)^2}{1 - (\omega/\omega_0)^2}.
\]

\[
\theta = \frac{m_2}{m_1}, \quad \omega_0^2 = \frac{k_2}{m_2}
\]

\[
m_{\text{st}} = m_1 + m_2
\]

Negative effective mass for

\[
1 < \left( \frac{\omega}{\omega_0} \right)^2 < 1 + \theta.
\]

Control bandgap frequencies

low frequency band gap: heavy & soft

Lead coated spheres + layer of silicone rubber in cubic array

very low frequency band gap and negative elastic constant due to dipole resonance

sub-wavelength band gap $\lambda \gg a$

negative effective modulus

unit cell = Helmholtz resonator

\[ \omega_0 \approx c \sqrt{S/L'} \sqrt{V} \]

V = cavity vol,
L' = neck length,
S = neck X-section

effective modulus:

\[ E_{\text{eff}}^{-1}(\omega) = E_0^{-1} \left[ 1 - \frac{F \omega_0^2}{\omega^2 - \omega_0^2 + i \Gamma \omega} \right] \]

negative effective modulus gives negative group velocity leading to group delay
2D sonic crystals with tailored properties

- uses multiple scattering theory (MST)

plane wave scattering from finite array of regularly spaced identical cylinders

equivalent material with radially varying sound speed

Sánchez-Dehesa, Torrent, L-W Cai (NJP 2009)

Martin et al. (APL 2010)
mechanism of sonic crystal focusing

dense solid cylinders in lighter fluid leads to effective medium with $v_{\text{eff}} < v_{\text{acoustic}}$ at the same time the effective impedance is not much different

“normal” lens effect
phononic crystals can lead to **negative refraction**

Positive refraction

![Positive refraction](image)

Negatives refraction

![Negative refraction](image)

- $\lambda \gg a$
- $\lambda \approx a$

potential for resolution beyond the diffraction limit

“perfect lens”
negative index for sound in water: fluid matrix NIM

Sukhovich, Jing, Page (PRB 2008)

phase matching to negative group velocity

ultrasonic negative index lens at 0.55 MHz
solid matrix negative index materials

Metal matrix
Morvan et al. APL 2010

Epoxy matrix
Croenne et al. PRB 2011

Tuning properties to efficiently couple with sound in water remains a challenge
phononic crystals in engineering applications

**Acoustic insulation**  
e.g. sound barriers using inexpensive materials  
(Sanchez-Dehesa et al. ‘10)

**Low frequency**  
vibration isolation, using internal resonators

**Waveguides, SAW filters**  
great potential, possible to fabricate, but radio-frequency devices still limited by energy loss  
(Lin et al. JAP ‘09)

**Negative index materials, lens**  
work well in theory, could provide super focusing, e.g. biomedical imaging  
- not yet practical: impedance mismatch, material limits  
(Croenne et al. ‘11)
minimalist sculpture by E. Sempere (1923-1985) in a Madrid park, was demonstrated to be a phononic crystal in 1995
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transformation acoustics
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Metatheory for metamaterials?
cloaking

engineering the impossible
background

Electromagnetic

equations are invariant under transformation of coordinates - old idea

- singular transformations yield cloaking

- 2D cloaking demonstrated at microwave frequency
  Schurig et al ('06)

Acoustic

-acoustic cloaking 2D and 3D models using singular transformation
  Chen & Chan ('07), Cummer et al ('07, '08)

-these models are restricted to anisotropic inertia
-

-general theory allows pentamode material (Norris ('08, '09)

practical demonstration of:
  broadband ultrasonic cloak (Zhang et al., PRL 2010)
  acoustic carpet cloak (Popa et al., PRL 2011)
broadband cloaking demonstration

S. Zhang, C. Xia and N. Fang (PRL 2011)

60 kHz

2D transmission line approach

uses sub-wavelength acoustic Helmholtz resonator

no cloak

with cloak
Acoustic carpet cloak in air

Popa, Zigoneanu, Cummer (PRL June 2011)

Idea of “carpet cloak”: cover the body so it looks like there is nothing there

theory/simulation

experiment at sonic frequency
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cloaking

transformation acoustics
1D acoustic cloak mirage/illusion

\[
\rho \quad K \\
\rho' \quad K' \quad c = \sqrt{\frac{K}{\rho}}
\]

“cloaked” region

\[\rho' c' = \rho c\]

\[\frac{b}{c'} = \frac{a}{c}\]

\[\rho' = \frac{a}{b} \rho\]

\[K' = \frac{b}{a} K\]

idea of transformation: geometrical mapping defines the material properties
same impedance/no reflection

same travel times

transformation $\xrightarrow{\text{material properties}}$

total mass is conserved

$\int \rho' \, dx = \int \rho \, dX$
scattering of waves

= reflection in all directions

much harder!
2D acoustic mirage

impedance
\[ \frac{\rho' c'}{\cos \theta'} = \frac{\rho c}{\cos \theta} \]

travel time
\[ \frac{b}{c' \cos \theta'} = \frac{a}{c \cos \theta} \]

Snell
\[ \frac{1}{c'} \sin \theta' = \frac{1}{c} \sin \theta \]

\[ \rho' = \frac{a}{b} \frac{\rho}{2 \cos^2 \theta} \left[ 1 + \sqrt{1 - \left(\frac{b}{a}\right)^2 \sin^2 2\theta} \right] \]

\[ K' = \frac{b}{a} K \]

Idea works at only one angle of incidence
finite transformation (deformation)

\[ x = \chi(X) \]

\[ F = \nabla_x x \]

\[ V^2 = FF^t \]

\[ J = \det F \]

Change of coordinates/Laplacian in new variables:

\[ \nabla^2_X p = J \text{ div } (J^{-1}V^2 \nabla p) \]
transformation of the acoustic wave equation

\[ \nabla^2_{\chi} p - \ddot{p} = 0 \quad \Rightarrow \quad J \text{ div } (J^{-1} V^2 \nabla p) - \ddot{p} = 0 \]

identical to

\[ K \text{ div } (\rho^{-1} \nabla p) - \ddot{p} = 0 \]

if

\[ K = J, \quad \rho = JV^{-2} \]

equation transforms if the density in the deformed (current) description is anisotropic

= idea behind transformation acoustics
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Metatheory for metamaterials?
anisotropic inertia?

standard acoustic wave equation
\[ \nabla^2 \rho - \ddot{\rho} = 0 \]

standard pressure constitutive relation
\[ \dot{\rho} = -K \text{ div } \mathbf{v} \]

momentum balance
\[ \rho \dot{\mathbf{v}} = -\nabla \rho \]

with anisotropic inertia tensor
\[ \rho \]

\[ \Rightarrow \]
\[ K \text{ div } (\rho^{-1} \nabla \rho) - \ddot{\rho} = 0 \]

modified acoustic wave equation

e.g. layered fluid exhibits anisotropic inertia

\[ \rho_2 = \langle \rho \rangle \]

\[ \rho_1 = \langle \rho^{-1} \rangle^{-1} \]
2D mirage using anisotropic inertia

\[ K = JK, \quad \rho = J \nabla^{-2} \rho \]

\[ J = \text{det} \mathbf{F}, \quad \nabla^2 = \mathbf{F} \mathbf{F}^t \]

\[ F_{ij} = \frac{\partial x_i}{\partial X_j} \]

\[ \mathbf{F} = \begin{bmatrix} 1 & 0 & \frac{b}{a} \end{bmatrix} \]

\[ K' = \frac{b}{a} K \]

\[ \rho_1 = \frac{b}{a} \rho, \quad \rho_2 = \frac{a}{b} \rho \]

works for all angles of incidence
Acoustic carpet cloak in air

How?

physical region is mapped into larger virtual one

Popa, Zigoneanu, Cummer (PRL June 2011)

layered scaffold of heavy plates

- has effect of slowing & curving the wave so that it appears to reflect from the flat surface

- curved 2D mirage device
- carpet cloak in water using curved steel plates

Liang & Li (APL 2011)
using transformation acoustics to design wave control devices

example: cylindrical-to-plane wave lens

Layman et al. (APL 2011 doi: 10.1063/1.3652914)

range of materials used
cylindrical source radiates as plane waves
anisotropy is a big part of cloaking

\[ \varepsilon_r = \mu_r = \frac{r - a}{r}, \quad a < r < b \]

\[ \varepsilon_\theta = \mu_\theta = \frac{r}{r - a} \]

\[ \varepsilon_z = \mu_z = \left( \frac{b}{b - a} \right)^2 \frac{r - a}{r} \]

extremely anisotropic

Metamaterial Electromagnetic Cloak at Microwave Frequencies  Schurig, et al., Science, 2006
aside: strange effects in cloaking - rays in a cloak

rays = deformed straight lines under transformation $F$

the wavefront bifurcates

the line through the singular point becomes a *split ray*

the “uncaustic”

ray bundle density $\rightarrow 0$
cylindrical anisotropic inertia

\[ \rho_\perp = \langle \rho^{-1} \rangle^{-1} \]

Momentum Balance

\[ \rho \frac{\partial \mathbf{v}}{\partial t} = -\nabla p \]

\[ [\rho] = \begin{bmatrix} \rho_r & 0 \\ 0 & \rho_\perp \end{bmatrix} \]

Torrent and Sanchez-Dehesa (2008)
homogenized cylinders

\[ \rho_1, c_1 \]
\[ \rho_2, c_2 \]

with anisotropic inertia

effective properties

\[ \rho_r, \rho_\theta, C_{\text{eff}} = \langle C \rangle \]

\[ C = K^{-1} \]

\[ \rho_r = \langle \rho \rangle \]
\[ \rho_\theta = \langle \rho^{-1} \rangle^{-1} \]

\[ \rho_r = \langle \rho^{-1} \rangle^{-1} \]
\[ \rho_\theta = \langle \rho \rangle \]
application: acoustic hyperlens

images sub-wavelength sources

Li et al. (Nature Materials 2009)
doi: 10.1038/nmat2561

\[ \rho_r = \langle \rho^{-1} \rangle^{-1} \]
\[ \rho_\theta = \langle \rho \rangle \]
cloaking: metafluid + hole

virtual / original

physical / transformed

$R = R_0$

$r = b$

transformation

$R = b$

$\rho$

$K$

physical

virtual
radially symmetric cloak

The perfect cloak has inner boundary equal to the image of a point.

In a near cloak it is a small hole.
three-fluid inertial cloak  

\[
\begin{pmatrix}
1 & 1 & 1 \\
\rho_1 & \rho_2 & \rho_3 \\
\rho_1^{-1} & \rho_2^{-1} & \rho_3^{-1}
\end{pmatrix}
\begin{pmatrix}
\phi_1 \\
\phi_2 \\
\phi_3
\end{pmatrix}
= 
\begin{pmatrix}
1 \\
\rho_r \\
\rho_\perp^{-1}
\end{pmatrix}
\]

\[\implies \phi = f_0 + \rho_r f_1 + \rho_\perp^{-1} f_2\]

effective compressibility:  

\[C_* = \alpha + \beta_1 \rho_r + \beta_2 \rho_\perp^{-1}\]

\[C_* = R'(\frac{R}{r})^{d-1}, \quad \rho_r = R'(\frac{r}{R})^{d-1}, \quad \rho_\perp^{-1} = R'(\frac{R}{r})^{d-3}\]

ODE for \(R(r)\)  \[\implies\] explicit transformation  \(r \rightarrow R(r)\)

Bottom line: can achieve exact transformation using only 3 distinct fluids mixed in different proportions
example: rigid scatterer with 3-fluid near-cloak

- can reduce scattering cross-section to less than 1%
- requires one very heavy fluid, one very light
- total mass of device is LARGE

- BUT – the types of fluid required are not available e.g. huge density + huge compressibility

\[ kr_0 = 3 \]
near cloaks made of inertial metafluids have very large mass

\[ m = \frac{3}{b^3 - r_0^3} \left( \frac{r_0^4}{R(r_0)} - \frac{b^4}{R(b)} \right) + \text{finite} \]

perfect cloak:

\[ R(r_0) = 0 \]

The perfect cloak has infinite mass

– regardless of the transformation (2D or 3D, radially symmetric or not)
How to make a cloak?

Acoustic “Invisibility” Cloaks Possible, Study Says

How to Make a Submarine Disappear

Acoustic metamaterials: Silence all around

How to Build an Acoustic Invisibility Cloak

e tc.

All inertial cloaks - infinitely massive

How to build an acoustic invisibility cloak

Metamaterials are all the rage in engineering and it’s not hard to see why. An entirely new class of substance with all kinds of exotic properties, metamaterials are being groomed as the building blocks of a new age of super lenses, agile anteneas and invisibility cloaks.

Until recently, though, all the focus had been on the way metamaterials can modify light. But what about other kinds of waves, such as sound? An acoustic version of this stuff would do the same for sound as it does for light—lenses, invisibility, and all.

At first glance, this sounds like a tricky ask. Metamaterials were first dreamed up by the Russian physicist Viktor Veselago, who imagined how a material might behave if its permittivity and permeability—factors that determine how a substance interacts with electric and magnetic fields—were both negative. Such materials never occur in nature, but have recently been constructed by engineers.

Making a metamaterial for sound means identifying the acoustic analogues to permittivity and permeability than working how to build a material in which they are both negative.

It turns out that the acoustic analogues in question are a material’s mass density and its elastic constant. And this week, a group of physicists at Wuhan University in China describes how it could be done.

Their proposed metamaterial is truly weird. It consists of a periodic array of rubber-coated gold spheres along with spheres of water containing air bubbles, all embedded within an epoxy resin.
Path to acoustic cloaking?

Transformation $\rightarrow$ radially anisotropic density $\rightarrow$ cloak $\rightarrow$ zero scattering

Making a fluid with strong density anisotropy is **difficult/impossible**.

Other issues: total mass required is very large

**Why density? Why not bulk modulus?**
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cloaking
pentamode materials
water as an elastic “solid”

elastic equation of motion  \( \text{div}\sigma = \rho \ddot{u} \)

\[
\sigma_{ij} = C_{ijkl} \varepsilon_{kl} \quad \rightarrow \quad \sigma = C \varepsilon
\]

Kelvin (1856)  
\[
C_{ijkl} = \sum_{\alpha=1}^{6} K_\alpha P_{ij}^\alpha P_{kl}^\alpha \quad \rightarrow \quad C = \sum_{\alpha=1}^{6} K_\alpha P^\alpha \otimes P^\alpha
\]

Acoustics  
\[
C_{ijkl} = K \delta_{ij} \delta_{kl} \quad \rightarrow \quad C = K \mathbf{I} \otimes \mathbf{I}
\]

Five of the eigen-stiffnesses are zero. Water is a \textbf{pentamodal elastic material}

(Milton & Cherkaev 1994) five (penta) easy modes

general pentamode form of stiffness:  
\[
C = K S \otimes S
\]

S is divergence free
pentamode material = transformed acoustic medium

\[
\begin{align*}
\text{constitutive relation} & \quad \text{momentum balance} \\
\sigma &= K \text{tr}(S\varepsilon) \ S \\
\rho \dot{v} &= \text{div} \ \sigma
\end{align*}
\]

rewrite in 'acoustic' form:

\[
\begin{align*}
\sigma &= -pS \\
\dot{p} &= -KS : \nabla v
\end{align*}
\]

pseudo-pressure \( p(x,t) \)

Use the PM compatibility condition

\[ \text{div} \ S = 0 \]

wave equation for the pseudo-pressure

\[
KS : \nabla \left( \rho^{-1} S \nabla p \right) - \ddot{p} = 0
\]

(was

\[
K \text{div} \left( \rho^{-1} \nabla p \right) - \ddot{p} = 0
\]

Can transform back to acoustic eq in \( X \) for arbitrary \( S \) as long as it satisfies

\[ \text{div} \ S = 0 \]
why can we have different types of transformed materials?

started with \( \nabla^2_X p - \ddot{p} = 0 \)

transformed coordinates \( \Rightarrow \quad J \text{ div } (J^{-1} \nabla^2 \nabla p) - \ddot{p} = 0 \)

interpreted as eq. of acoustic fluid with anisotropic inertia \( K = J, \quad \rho = J \nu^{-2} \)

but we did not change the meaning of the pressure (or particle displacement)

relax this and redefine a “transformed” displacement

\( U = A u \)

then \( A \) is related to appearance of the matrix \( S \) in the pentamode material
mechanical behavior of **pentamode materials**

\[ C_{ijkl} = K Q_{ij} Q_{kl} \]

a single type of stress (and strain)

- generalize hydrostatic stress and volumetric strain of an **acoustic fluid**

Static equilibrium of a block ... under gravity

“microstructure”
The metafluid is any material of the form *

\[
K = J, \quad C = KS \otimes S, \quad \rho = JSV^{-2}S,
\]

(Norris ’08, ’09)

- \( J \) (Jacobian) and \( V \) (metric) are defined by the (arbitrary) transformation
- \( S \) is \textit{any} definite symmetric tensor satisfying \( \text{div} \ S = 0 \)

anisotropic inertia is the special case \( S = I \)

\( S \) introduces additional degrees of freedom, non-uniqueness (unlike EM)

1) most general type of acoustic metafluid is \textbf{PM} with anisotropic inertia

2) can get \textit{isotropic density} by choosing \( S \) to make it so

*it can be even more general: \( S \) does \textit{not} have to be symmetric - leading to nonsymmetric stress
mirages

using pentamode material

\[
\begin{bmatrix}
1 & 0 \\
0 & \frac{b}{a}
\end{bmatrix}
\]

\[ \mathbf{F} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{b}{a} \end{bmatrix} \]

\[ \mathbf{\rho}' = J^{-1} \mathbf{\rho} \mathbf{I}, \quad \mathbf{K}' = J \mathbf{K}, \quad \mathbf{S} = J^{-1} \mathbf{V}, \]

\[ \Rightarrow \quad \frac{b}{a} \mathbf{K}, \quad \frac{a}{b} \mathbf{\rho}, \quad \mathbf{S} = \begin{bmatrix} \frac{a}{b} & 0 \\ 0 & 1 \end{bmatrix} \]

Isotropic density with total mass conserved
PM supports biaxial stress
pentamode material transformation

- looks a lot like the inertial anisotropy transformation:

\[ \rho = R' \left( \frac{R}{r} \right)^{d-1}, \quad K_r = \frac{1}{R'} \left( \frac{R}{r} \right)^{d-1}, \quad K_\perp = R' \left( \frac{R}{r} \right)^{d-3} \]

- gives the same wave steering effects
- but the **mechanics** is completely different
freedom to choose S means the cloak material has isotropic inertia
e.g. for any radially symmetric transformation

Important PM property - like 1D (used later):

$$\int_{\Omega} d\nu J^{-1} \rho = \int_{\Omega_0} d\nu_0 \rho_0 = m(\Omega_0)$$

The total mass of the cloak is conserved under the transformation

a non-radially symmetric cloak with isotropic density (& finite mass of course)
simulation: acoustic scattering from a steel sphere

no cloak  layered pentamode cloak
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Metatheory for metamaterials?
design of pentamode materials
possible designs of acoustic cloaking devices

1) **anisotropic inertia** via fine layering
   - highly constrained
   - large mass issues
   - excessive material properties

2) **Pentamode Material**
   - isotropic inertia
   - solid based
   - microstructure
   - use ideas from composite materials

*Idea: unit cell supports only stress $S$*

force/truss structure for 2D cloak
Pentamode Material (possible) microstructures

Fluid only supports stress \(-p\)

PM supports only stress \(S\)

Idea: create unit cell that supports \(S\) (elements indicate forces)
isotropic and anisotropic networks

**Isotropy**

$4V_0 = 6\sqrt{3}a^2$

2a

$I = \sum_{j=0}^{d} a_j \otimes a_j$

$a_0 = a(0, -1), \quad a_1 = a(\frac{\sqrt{3}}{2}, \frac{1}{2})$

$a_2 = a(-\frac{\sqrt{3}}{2}, \frac{1}{2}), \quad a = \sqrt{\frac{2}{3}}$

**Anisotropy**

$S = S^{1/2} IS^{1/2}$

$S = \sum_{j=0}^{d} a'_j \otimes a'_j$

$a'_j = S^{1/2} a_j$
a generic material?

some desired/necessary properties:

• **transparent** to underwater sound in its “base state”

• **density** of water in its base state  (conservation of mass property)

• effective **bulk modulus** of water in base state

• “**deformable**” to achieve anisotropy

• **maximize** potential cloaked space
pentamode material (PM) microstructure possibilities

isotropic

anisotropic
cellular foam structures

extensional and bending compliances (inverse of stiffness)

\[ M_a = \frac{a^2}{EA}, \quad N_a = \frac{a^2}{EI} \]

Effective elastic moduli

\[ C = \begin{pmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{pmatrix} = \begin{pmatrix} \frac{E_1}{1-\nu_1^2} & \frac{\nu_1^2 E_2}{1-\nu_1^2 \nu_2^2} & 0 \\ \frac{\nu_1^2 E_1}{1-\nu_1^2 \nu_2^2} & \frac{E_2}{1-\nu_1^2 \nu_2^2} & 0 \\ 0 & 0 & G_{12} \end{pmatrix} \]

\begin{align*}
E_1^* &= \frac{l \sin \theta}{2b(h + l \cos \theta)(N_1^2 \cos^2 \theta + M_1 \sin^2 \theta)} \\
E_2^* &= \frac{l \sin \theta}{2b(l \cos \theta)(N_1^2 \sin^2 \theta + M_1 \cos^2 \theta + 2M_h)} \\
\nu_1^* &= \frac{l \sin \theta}{l \sin \theta(N_1^2 \cos^2 \theta + M_1 \sin^2 \theta)} \\
\nu_2^* &= \frac{l \sin \theta}{l \sin \theta(N_1^2 \sin^2 \theta + M_1 \cos^2 \theta + 2M_h)} \\
G_{12} &= \frac{2b((h^2 N_1^2 + 2l^2 N_h^2) \sin^2 \theta + M_1 (l + h \cos \theta)^2)}{}
\end{align*}

(Kim & Hassani 2003)
cellular foam as a Pentamode Material

thin members have small bending stiffness, leading to (approximately)

$$C = \begin{pmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{pmatrix} = C_0 \begin{pmatrix} \alpha & 1 & 0 \\ 1 & \frac{1}{\alpha} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_0 = \frac{\sin \theta \cos \theta}{2b(M_l + 2M_h \sin^2 \theta)}$$

$$\alpha = \frac{l \cos^2 \theta}{(h + l \sin \theta) \sin \theta}$$

acoustic fluid: \( c = \begin{pmatrix} K & K & 0 \\ K & K & 0 \\ 0 & 0 & 0 \end{pmatrix} \)  \( \rightarrow \) Cellular structure has PM form with unique stress

\( \sigma = -p \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \)

bottom line: cellular foam-like structure with thin members can provide the PM stiffness

\( C_{66} \neq 0 \) but is small, \( C_{66}/C_0 = O\left(\frac{M}{N}\right) = \frac{\text{bending stiffness}}{\text{extensional stiffness}} \)

with relatively small shear rigidity

thin members = more potential cloaking space

........... stiff, dense
Metal Water

Idea: make water from metal foam

e.g. start with block of Al, remove solid metal to get:

a) effective density of water
b) effective isotropic elasticity with bulk modulus of water, small shear modulus
MW prototypes

- Have manufactured prototypes from Aluminum plate stock using a water-jet process.
- In-water transmission testing of the full part completed.

Bulk modulus = 2.25 Gpa
Density = 1000 kg/m^3

Shear modulus = 0.065 GPa
FEM

**islands**: inertial role only - denser is better

**isthmuses**: elastic role only - stiffer is better

\[ M_a = \int_0^{a/2} \frac{dx}{EA}, \quad N_a = \int_0^{a/2} \frac{x^2 \, dx}{EI} \]

metal
Eigenanalysis of the Metal Water design indicates shear-related modes.

As expected, since the design cannot be and is no perfectly Pentamode.

Shear modes are high-k, low-frequency, easily damped, not expected to affect test results.

Transient explicit analysis of the Metal Water design indicates that the shear-related deformation modes do not have a strong effect on wave propagation, when a sample is ensonified using a plane wave.

FLEX simulations
(Weidlinger Associates)
Metal Water $\rightarrow$ cloak

New metafluid = radially compressed version of metallic water
   same total mass as uncompressed region

Conservation of cloaked space
conservation of mass = conservation of cloaked space

Heavy metal preferred
Metal Water offers the potential of solid structures that

• mimic the acoustic properties of water (isotropic)
• enable transformation acoustics (anisotropic)

using the long wavelength homogenization properties of structured metallic foams as Pentamode materials

Other uses?
Metal Water: application to negative index materials

Dispersion curve in the first Brillouin zone

material properties appear to match with sound in water

Goal: NIM lens with MW using negative group velocity branches
Introduction: metamaterials

Phononic crystals – engineering the band gap waves in periodic systems
examples of phononic crystals

Acoustic cloaking – engineering the impossible transformation acoustics
1D, 2D, cylindrical inertial materials
Pentamode materials cloaking elastic waves

Metatheory for metamaterials?
cloaking of elastic waves
Miraging and cloaking, work in principle - but require materials that are not “elastic”

- Density can always be made isotropic
- Stress is usually not symmetric, Cosserat materials are necessary
- Normal elastic materials can provide approximate cloaking
elastic transformation theory

- transformation

\[ \mathbf{X} \rightarrow \mathbf{x} \quad F_{iJ} = \frac{\partial x_i}{\partial X_J} \]

- as in acoustics, the materials are not unique. They can be characterized by how the displacement transforms

\[ \mathbf{U} \rightarrow \mathbf{u} = \mathbf{A} \mathbf{U} \]

- 2 parameters: \( \mathbf{F}, \mathbf{A} \)

- material generally of Willis form, with properties including matrix density that are functions of frequency

- BUT constant isotropic density if \( \mathbf{A} = \mathbf{I} \) then moduli of Cosserat form

\[
\rho^{\text{eff}} = J^{-1} \rho_0, \quad C_{ijkl}^{\text{eff}} = J^{-1} F_{iI} F_{kK} C_{IjKl}^{(0)}
\]

(ANN, ALS 2011)
elastic transformation $\rightarrow$ unique nonlinear materials

\[ \rho^{\text{eff}} = J^{-1} \rho_0, \quad C_{ijkl}^{\text{eff}} = J^{-1} F_{ik} F_{jk} C_{ijkl}^{(0)} \]

(ANN, ALS 2011)

transformed elastic moduli are tangent moduli for a hyperelastic material with strain energy under a state of (pre)stress

\[ W = \frac{1}{2} (U_{j\alpha} - \delta_{j\alpha}) (U_{l\beta} - \delta_{l\beta}) C_{\alpha j \beta l}^{(0)} \]

\[ \sigma_{ij}^{\text{pre}} = J^{-1} F_{i\alpha} (F_{l\beta} - \delta_{l\beta}) C_{j\alpha \beta l}^{(0)} \]

- equilibrium of the pre-stress constrains the transformation \( F \) to satisfy

\[ C_{j\alpha \beta l}^{(0)} x_{l,\alpha \beta} = 0 \]

- for acoustics, this includes all radially symmetric (d=2,3) transformations
cloaking in elasticity

this means elastic waves can be cloaked using a material with a **unique finite strain energy function**

expand small holes to finite size: the deformed solid has small-on-large moduli *exactly* those required from the transformation

**ex. SH waves**

bulk material          initial hole          expanded

line source
- rigid cylinder
- no cloak

pressurize small hole to size of cylinder, and insert rigid body

scattering from original hole

neo-Hookean strain energy function

\[ \mathcal{W} = \frac{\mu}{2}(\lambda^2 + \lambda^2 + \lambda^2 - 3) \]

\[ \mu \] = original shear modulus
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Metatheory for metamaterials?
A metatheory for metamaterials?

**Phononic crystals:**

Homogenization of periodic systems at finite frequency & wavelength leads to effective material properties consistent with the

**Willis constitutive model:**

\[
\begin{align*}
\sigma^{\text{eff}} &= \nabla \cdot \sigma^{\text{eff}} \\
\left(\begin{array}{c}
\sigma^{\text{eff}} \\
\rho^{\text{eff}}
\end{array}\right) &= \left(\begin{array}{cc}
C^{\text{eff}} & S^{\text{eff}} \\
-S^{\text{eff}} & \rho^{\text{eff}}
\end{array}\right) \left(\begin{array}{c}
\varepsilon^{\text{eff}} \\
\dot{\varepsilon}^{\text{eff}}
\end{array}\right)
\end{align*}
\]


**Acoustic and elastic cloaking:**

The most general constitutive model that is invariant under “transformation” is the **Willis constitutive model.** (Milton, Briane, Willis, 2007, ANN 2011)
in closing

metamaterials, especially cloaking devices, combine a rich mixture of topics

• wave mechanics/physics
• continuum mechanics
• differential geometry
• anisotropic elasticity
• finite elasticity
• materials science
• fabrication issues
• computational methods
• etc.
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