PROOF OF THE FERMAT'S LAST THEOREM

Andrew H. Warren

Originally submitted to American Mathematical Society on February 16, 1991. Not rejected – no flaws were found. Not accepted – the referees could not grasp the key concept. The author hopes that improved presentation and better graphics will make the understanding of the concept easier to grasp.

INTRODUCTION

The Theorem states that if **L**, **M**, **N** and **n** are all integers then for **n> 2**

$$\mathbf{N}^n \neq \mathbf{M}^n + \mathbf{L}^n$$

The objective of this derivation is to prove that this is the case.

Integers raised to an integer power **n** will be represented by **n**-dimensional double-subscripted arrays. The arrays will, in turn, be represented by geometrical figures made up of units, which will be **2n**-faceted geometrical entities. A complete assembly of units representing an **n**-dimensional array will be referred to, in the geometrical sense, throughout the text as a super-cube even though "super-cube" may be n-dimensional with $\infty > n \geq 3$. A super-cube is made-up of multifaceted unit cubes. The faces of a unit cube, which are in contact with the faces of other unit cubes will be referred to as the inner faces. The faces on the boundaries of a super-cube, not in contact with the other faces, will be referred to as the outer faces. The proof of the Fermat's Last Theorem will be derived utilizing such a geometrical representation of integer numbers raised to an integer power. The leading thought throughout the derivation is illustrated in Fig. 1. When one super-cube made up of unit cubes is subtracted from a larger super-cube there are too many outer faces for the remaining unit cubes as compared to the number of outer faces for a perfect super-cube.

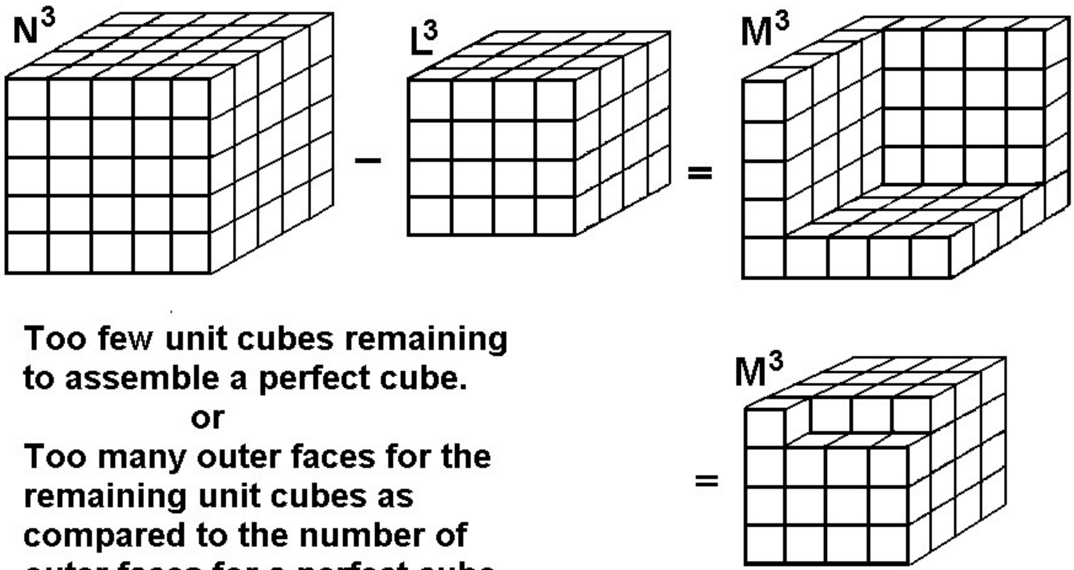

Figure 1. Geometrical representation of the arithmetic operations on integers.

1. ASSUMPTIONS

n is a positive integer and an exponent of a number raised to a power

$$\infty > \mathbf{n} \geq \mathbf{1} \tag{1}$$

L, **M** and **N** are positive numbers where:

$$\infty > \mathbf{L} \geq \mathbf{1}$$
$$\infty > \mathbf{M} \geq \mathbf{1} \tag{2}$$
$$\infty > \mathbf{N} \geq \mathbf{1}$$

The relative magnitude of **L**, **M** and **N** is:

$$\mathbf{N} > \mathbf{M} \geq \mathbf{L} \tag{3}$$

L, N, $\mathbf{L^n}$, $\mathbf{M^n}$ and $\mathbf{N^n}$ are all integers.

It will be investigated that if equation (4), below,

$$\mathbf{N^n - L^n = M^n} \tag{4}$$

is to be satisfied, for $\mathbf{n > 2}$, can **M** be an integer.

By definition, an integer is a collection of units, where a unit is synonymous with **1**. The letter **U**, where $\mathbf{U \equiv 1}$, will be used to denote a unit.

Note: **M** will be assumed to be an integer until it is shown that it cannot be an integer.

2. ARRAY REPRESENTATION OF INTEGERS RAISED TO AN INTEGER POWER

It will be illustrated in this section how integers raised to an integer power are represented by double-subscripted arrays and then as assemblies of geometrical units.

In the one-dimensional case an integer $\mathbf{N^1}$ can be presented as a double subscripted array and also in a geometrical representation in a following way:

$$N^1 = U_{0,1} \;\; U_{1,2} \;\; U_{2,3} \;\; U_{3,4} \;\; \ldots\ldots U_{N-2,N-1} \;\; U_{N-1,N} \tag{8a}$$

Figure 1a. An integer, represented as a one-dimensional array.

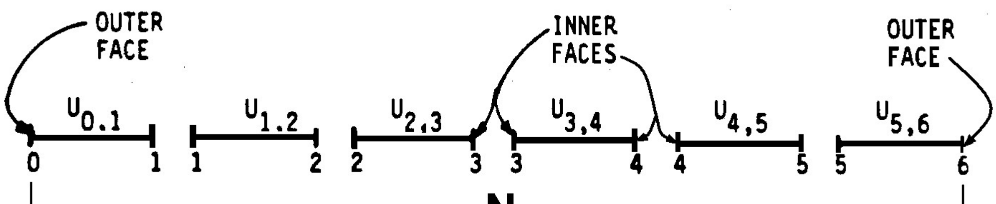

Figure 1b. A geometrical representation of a one-dimensional array - exploded view

The subscripts represent sides of units. The sides (subscripts) in contact with other sides are called the ***inner subscripts***. The sides (subscripts) not in contact with the other sides are called the ***outer subscripts***, i.e. subscripts that are not repeated in a double subscripted array are the outer subscripts.

In a string of subscripts: [0, 1, 1, 2, 2, ...N-2, N-2, N-1, N-1, N] **0** and **N** are the outer subscripts.

NOTE: “faces” or “sides” of a unit will be synonymous with “subscripts”. A number of faces form a surface. Faces are on each side of a unit in any of the **n**-directions, i.e. on the positive side and the negative side of a unit.

In the two-dimensional case an integer $\mathbf{N^2}$ can be presented in a geometrical representation (Figs. 2a and 2b) and also as a double subscripted array (Fig. 2c) in a following way:

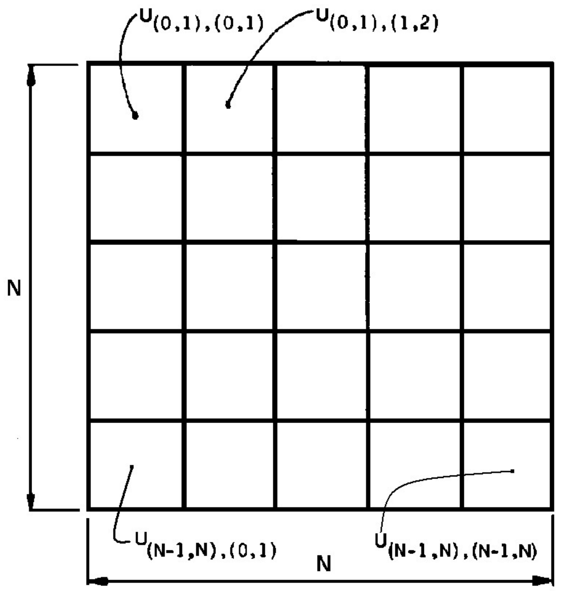

Figure 2a. A geometrical representation of a two-dimensional array

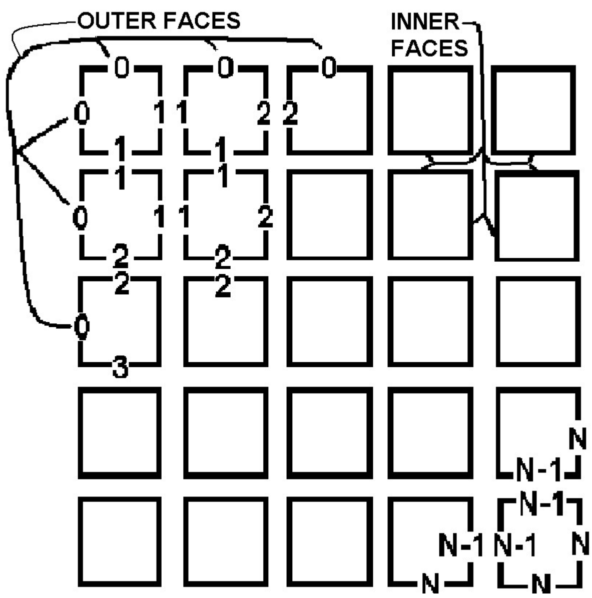

Figure 2b. A geometrical representation of a two-dimensional array.- exploded view

$$
\begin{array}{llll}
U_{(0,1),(0,1)} & U_{(0,1),(1,2)} \;\ldots\ldots & U_{(0,1),(N-2,N-1)} & U_{(0,1),(N-1,N)} \\
U_{(1,2),(0,1)} & U_{(1,2),(1,2)} \;\cdots & U_{(1,2),(N-2,N-1)} & U_{(1,2),(N-1,N)} \\
U_{(2,3),(0,1)} & U_{(2,3),(1,2)} \;\cdots\cdots & \cdots\cdots & U_{(2,3),(N-1,N)} \\
\ldots & . \quad \ldots\ldots & \ldots\ldots\ldots\ldots & \\
U_{(N-2,N-1),(0,1)} \;\cdots\cdots & & \cdots\cdots & U_{(3,4),(N-1,N)} \\
U_{(N-1,N),(0,1)} & U_{(N-1,N),(1,2)} \ldots & U_{(N-1,N),(N-2,N-1)} & U_{(N-1,N),(N-1,N)}
\end{array}
\qquad \mathbf{(8b)}
$$

Figure 2c. An integer, N^2, represented as a two-dimensional array.

In a three-dimensional space each subscripted unit of an array representing $\mathbf{N^3}$ will have three pairs of subscripts as follows:

$$U_{(p-1,p),(q-1,q),(r-1,r)} \qquad \mathbf{(8c)}$$

where **p**, **q** and **r** are integers with values between **0** and **N**, inclusive.

Similarly, in a **n**-dimensional space each unit in an array representing an integer $\mathbf{N^n}$ will have **n** pairs of subscripts:

$$U_{(p-1,p),(q-1,q),(r-1,r),\ldots\ldots\ldots\ldots(s-1,s)} \qquad \mathbf{(8c)}$$

where **p**, **q**, **r** and **s** are integers with values between **0** and **N**, inclusive.

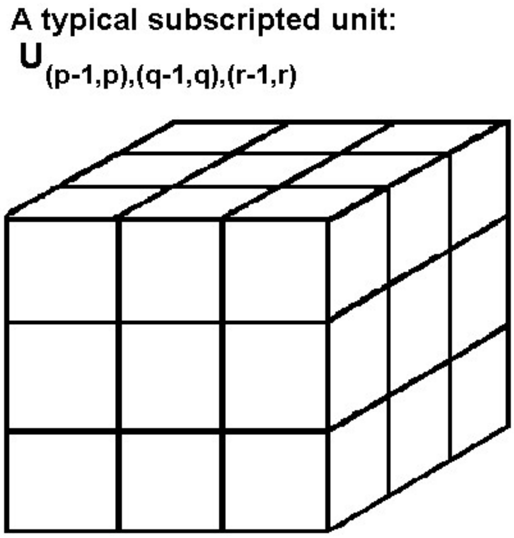

Figure 3a. A geometrical representation of a three-dimensional array

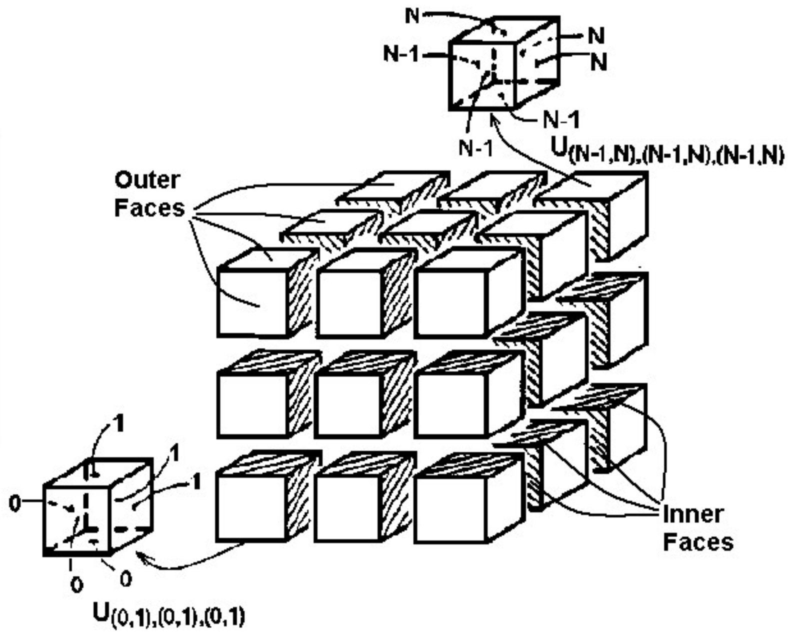

Figure 3b. A geometrical representation of a three-dimensional array.- exploded view

3. CHARACTERISTIC QUALITIES OF AN INTEGER RAISED TO AN INTEGER POWER n

In this section the qualities characteristic to an integer raised to an integer power **n** represented by an n-dimensional double-subscripted array will be defined.

$$\text{The total number of subscripts in an array} = 2\,n\,N^n \qquad \textbf{(10a)}$$

$$\text{Number of the outer subscripts} = 2nN^{n-1} \qquad \textbf{(10b)}$$

$$\text{Number of the inner subscripts} = 2\,n\,N^n - 2\,n\;N^{n-1}$$

$$= 2\,n\;N^{n-1}(N-1) \qquad \textbf{(10c)}$$

Expressions **(10a)**, **(10b)** and **(10c)** can be listed as characteristic for an integer raised to the **n** power.

Each unit has a positive and a negative face (or subscript) along each of the **n** directions. Therefore, the total number of subscripts (faces) in an **n**-dimensional array representing an integer raised to a power **n** is equal to the total number of units, $\mathbf{N^n}$, multiplied by the number of faces per unit - **2 n**.

The outer subscripts (faces) are **0** and **N**. Between **0** and **N**, inclusive, there are **N+1** different subscript values. There are $\mathbf{N^{n-1}}$ subscripts for each subscript value on the positive sides of the units and the same quantity of subscripts on the negative sides of the units. Consequently, there are $\mathbf{2N^{n-1}}$. subscripts for each subscript value for all the subscripts, except for **0** and **N**. The **0** subscripts represent the outer faces only on the negative sides of the super-cube (see Fig. 5) and the **N** subscripts represent the outer faces only on the positive side of the super-cube. For this reason there are $\mathbf{N^{n-1}}$ of the **0** subscripts and $\mathbf{N^{n-1}}$ of the **N** subscripts. Since the **0** and **N** subscripts are the outer subscripts, the total quantity of the outer subscripts is $\mathbf{2N^{n-1}}$.

4. **GEOMETRY – ALGEBRA RELATIONSHIP**

The entire derivation will be illustrated on cubes, i.e. **n=3** for easier visualization of the process. The expressions in the entire derivation will be applicable for **$n \geq 3$.**

A super-cube is defined as a geometric assembly of unit "cubes" representing in a geometrical sense a double subscripted array, which, in turn, represents an integer raised to an integer power **n**. A unit cube will denote in a geometrical sense a 2n-faceted entity – a unit cell of a super-cube. A unit cube is numerically synonymous with **1**.

5. CONDITIONS WHICH DETERMINE WHETHER M IS AN INTEGER OR NOT.

It has been already assumed, in the first section, that **L, N, L^n, M^n** and **N^n** are integers. According to section 3, a condition necessary to assure that **M** is an integer requires that the quantity of the outer faces corresponding to the array representing **M^n** is equal to the number evaluated by expression **(10b)** above, i.e. **$2\,n\,M^{n-1}$**. Consequently:

If the sum of the outer faces of an assembly remaining after the **L^n**-super-cube was subtracted from the **N^n**-super-cube is equal to **$2\,n\,M^{n-1}$** then **M** is an integer (units of **M^n** form a perfect "super-cube")

M is not an integer (units of **M^n** form an incomplete "super-cube") if the sum of the outer faces of the re-combined assembly remaining after **L^n**-super-cube was subtracted from the **N^n**-super-cube is not equal to **$2nM^{n-1}$**.

The above conclusions can be summarized:

If **$M^n = N^n - L^n$** and **M** is an integer then **$Q(M^n)$ = sum of M^n outer faces = $2\,n\,M^{n-1}$** **(11)**

If **$Q(M^n) \neq 2\,n\,M^{n-1}$** then M is not an integer. **(12)**

NOTE: It is assumed that the unit "cubes" of the investigated arrays are arranged in such a way as to form the smallest possible surface area.

Ultimately, it is intended to show that after the subtraction of **L^n** from **N^n**, the super-cube representing an array which represents **M^n** will have too many outer faces for its number of units. That is, it will be demonstrated that the remaining quantity of the outer faces of the super-cube representing **M^n** is greater than **$2n\,M^{n-1}$**.

6. DETERMINATION OF THE QUANTITY OF THE OUTER FACES PER UNIT CUBE

Each super-cube (whether **N^n**, **L^n** or **M^n**) is made-up of an assembly of unit cubes and is encompassed by an outer surface. The outer surface per unit cube is equal to the total outer surface (Expression **(10b)**) divided by the total number of unit cubes in a given super-cube (Expression **(10a)**). That is, in case of **N^n** the outer surface per unit cube will be:

$$2\,n\,N^{n-1} \;/\; N^n = 2n / N \qquad \textbf{(13)}$$

It is to be noted that the amount of outer surface per unit cube changes with the size of the super-cube as illustrated in Fig. 4. That is:

$$\text{Outer surface per unit cube} = \frac{2n}{N},\ \frac{2n}{N-1},\ \frac{2n}{N-2},\ \ldots\ \text{to}\ \frac{2n}{M} \qquad \textbf{(13a)}$$

One half of this outer surface will be used to cover the sides of the super-cube facing the positive n-directions (i.e., $+n_1$, $+n_2$, $+n_3$ ....$+n_n$). The other half of this outer surface will be used to cover the sides of the super-cube facing the negative n-directions (i.e., $-n_1$, $-n_2$, $-n_3$.......$-n_n$). (see Figure 4).

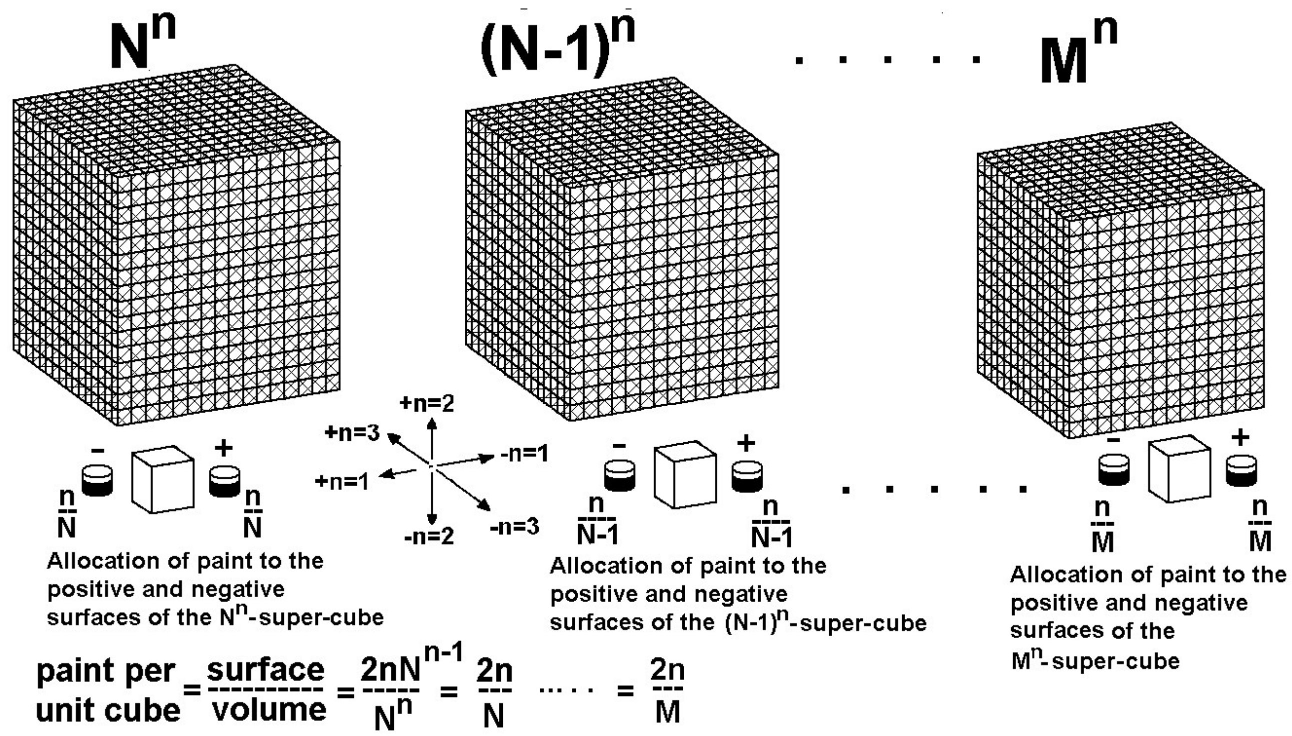

Figure 4. Allocation of outer surface (paint) per unit cube

7. DETERMINE THE MAXIMUM QUANTITY OF THE OUTER FACES TO BE REMOVED

In this segment of derivation the objective is to establish what is the maximum quantity of the outer surface that can be removed from the $\mathbf{N^n}$-super-cube when the $\mathbf{L^n}$-super-cube is subtracted from the $\mathbf{N^n}$–super-cube.

The process of subtraction of $\mathbf{L^n}$ from $\mathbf{N^n}$ resulting in $\mathbf{M^n}$ will be performed in a particular way.

It can be demonstrated – see Fig. 5 – that the $\mathbf{L^n}$-super-cube is encompassed by the $\mathbf{M^n}$-super-cube and in turn the $\mathbf{M^n}$-super-cube is encompassed by the $\mathbf{N^n}$-super-cube.

Consequently, as illustrated in Figures 5 and 6, when $\mathbf{L^n}$ is being subtracted from $\mathbf{N^n}$, it is, at the same time, subtracted from $\mathbf{M^n}$. Where $\mathbf{M^n}$ is the final super-cube after the subtraction process.

The process of subtracting the $\mathbf{L^n}$-super-cube from the $\mathbf{N^n}$-super-cube, is performed in two stages (see Fig. 6). First, the unit cubes are removed from the $\mathbf{L^n}$- super-cube space of the $\mathbf{N^n}$-super-cube. In the second stage the unit cubes from the $\mathbf{(N^n}$-$\mathbf{M^n)}$-layer fill-in the space left in the $\mathbf{N^n}$-super-cube from where the $\mathbf{L^n}$-super-cube was removed.

As observed already, when the $\mathbf{L^n}$-super-cube is being removed (subtracted) form the $\mathbf{N^n}$-super-cube it is simultaneously subtracted from the $\mathbf{M^n}$-super-cube. If we call the outer surface paint, the unit cubes of the subtracted $\mathbf{L^n}$-super-cube remove their paint contributions only from the side of the $\mathbf{M^n}$-super-cube facing the negative directions. ***The paint allocated to the sides of M^n-super-cube facing the positive n-directions must remain in place if the M^n-super-cube is to remain painted***.

The paint contributions of the unit cubes to the super-cube outer surface depend on the super-cube size and are defined by Exp. **(13a)**. Consequently, when the $\mathbf{L^n}$-super-cube is subtracted from the $\mathbf{N^n}$-super-cube the amount of paint removed is defined by Exp. **(13b)**, below:

$$\text{Amount of paint per unit cube which is being removed} = \frac{n}{\text{(current size of large super-cube)}} = \frac{n}{N},\ \frac{n}{N-1},\ \frac{n}{N-2},\ \ldots\ldots\ \text{to}\ \frac{n}{M} \qquad \mathbf{(13b)}$$

It follows from (13b) that the largest amount of paint ***per unit cube*** that can be removed from the $\mathbf{N^n}$-super-cube is **n/M**.

Consequently, it will be safe to state that the total paint removed from the $\mathbf{N^n}$-super-cube together with the $\mathbf{L^n}$ unit cubes is not greater than $\mathbf{L^n(n/M)}$.

The $\mathbf{(N^n\text{-}M^n)}$-layer) has the same quantity of the unit cubes and the same quantity of the outer faces as the $\mathbf{L^n}$-super-cube. Consequently, after the $\mathbf{L^n}$-super-cube is removed from the $\mathbf{N^n}$-super-cube, the unit cubes from the $\mathbf{(N^n\text{-}M^n)}$-layer are moved into the $\mathbf{L^n}$-space with precisely the amount of paint that was removed (see Fig. 7).

At the end of this process the space of the $\mathbf{N^n}$-super-cube where the $\mathbf{L^n}$-super-cube originally resided is filled-in and painted.

Since the largest possible quantity of the outer faces was removed, the paint remaining after the subtraction of $\mathbf{L^n}$ from $\mathbf{N^n}$ represents the minimum paint (outer surface) allocated for painting of $\mathbf{M^n}$-super-cube. Therefore, the minimum possible outer surface area remaining after the subtraction of $\mathbf{L^n}$ from $\mathbf{N^n}$ is:

$$2*n*N^{n-1} - L^n(n/M) \qquad \textbf{(14)}$$

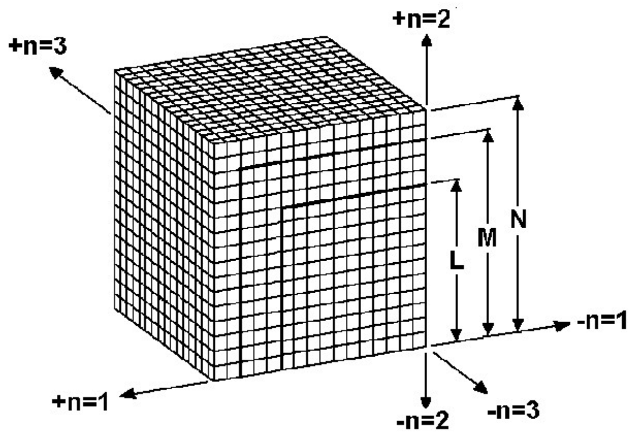

Figure 5. A three-dimensional example illustrating the placement of the cubes representing integers L^n, M^n and N^n

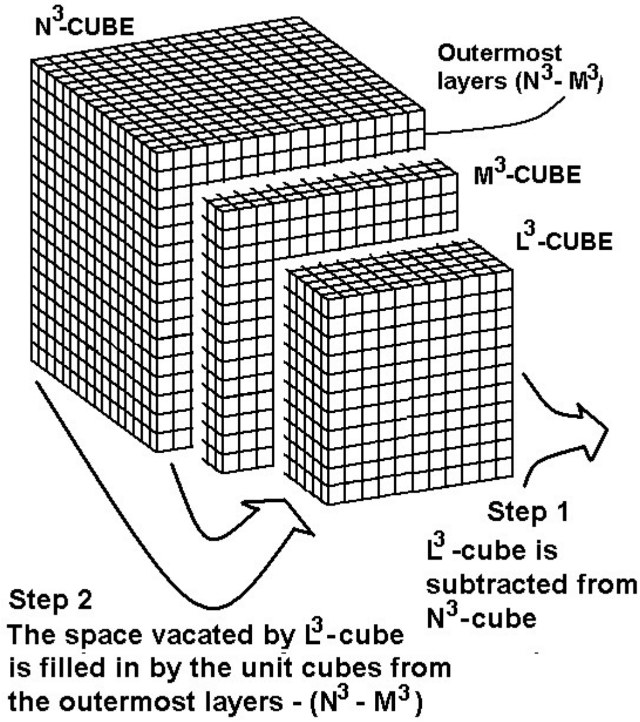

Figure 6. Two-step subtraction and re-combination process

8. THE SUMMARY OF A LOGICAL ARGUMENT (SEE FIGURE 7).

$\mathbf{L^n}$-super-cube is subtracted simultaneously from, say, the sides of $\mathbf{N^n}$-super-cube and $\mathbf{M^n}$-super-cube facing the negative n-directions.

*As stated earlier, it is assumed that **M** is an integer until it is proved that it is not.*

Since $\mathbf{M^n}$-super-cube is to be the final super-cube, following Exp. (13a) **2n/M** is the maximum outer surface per unit cube (whether in $\mathbf{N^n}$ or $\mathbf{M^n}$). A quantity of **n/M** of each unit cube allocated to the side of $\mathbf{M^n}$-super-cube facing the negative **n**-directions and **n/M** allocated to the side of $\mathbf{M^n}$-super-cube facing the positive **n**-directions.

The maximum outer surface that can be removed from the $\mathbf{M^n}$-super-cube (and the $\mathbf{N^n}$-super-cube) is $\mathbf{L^n\ n/M}$. That is, only the outer faces allocated to the unit cubes of $\mathbf{L^n}$-super-cube facing the negative **n**-directions are removed. More outer faces cannot be removed to assure that the side of $\mathbf{M^n}$-super-cube facing the positive **n**-directions remains completely painted after the process of subtraction and re-combination is completed.

Also, we cannot subtract more outer faces because the quantity of the outer faces which remain after the subtraction must perfectly match the outer surface corresponding to $\mathbf{M^n}$ array.

What remains is supposed to be $\mathbf{M^n}$ and have the outer surface of $\mathbf{M^n}$,-super-cube. As stated earlier, the unit cubes of the $\mathbf{(N^n\text{-}M^n)}$-layer must have a total of $\mathbf{L^n n/M}$ of the outer faces.

When the unit cubes of the $\mathbf{(N^n\text{-}M^n)}$-layer are moved into the hole left in $\mathbf{M^n}$-super-cube the entire surface of $\mathbf{M^n}$super-cube is completely painted. At the conclusion of this process the number of outer faces remaining is: $\mathbf{2nN^{n-1} - L^n\ n/M}$ The entire process is illustrated on Figure 7, below.

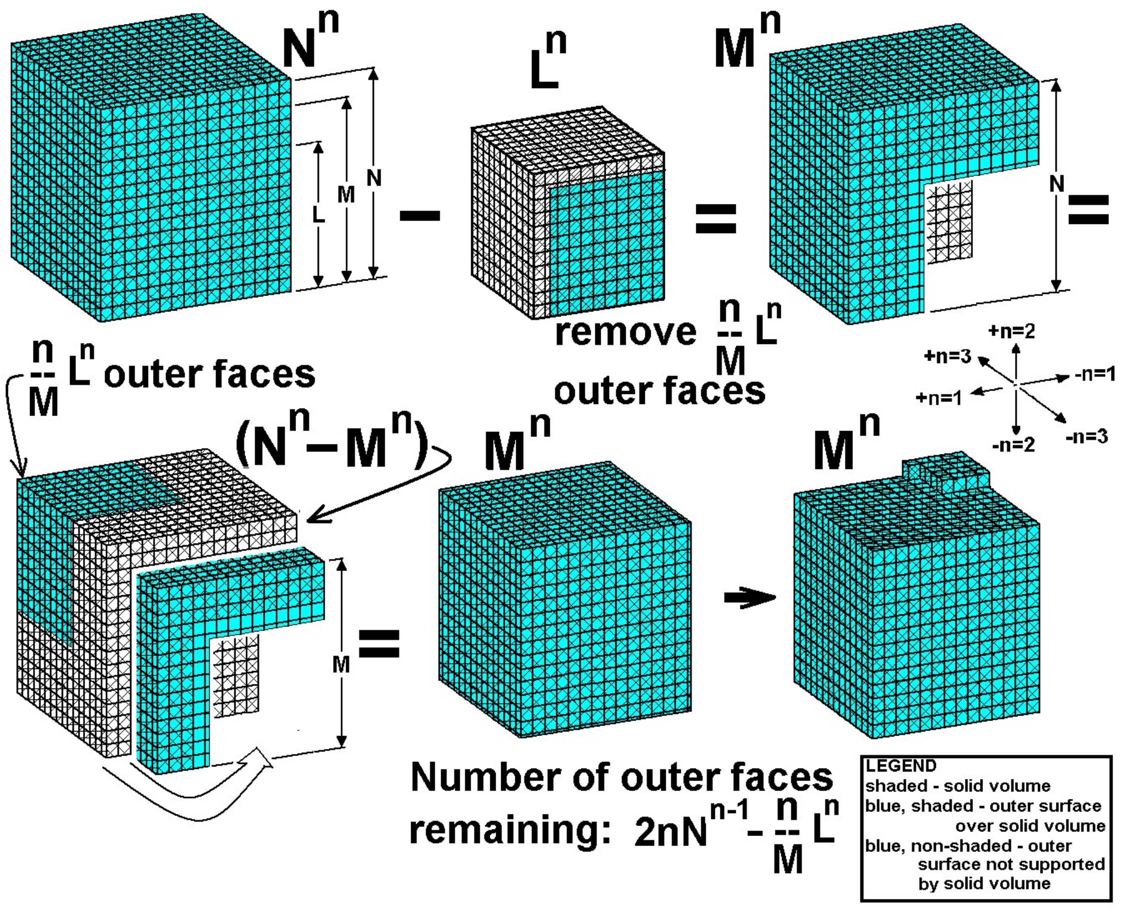

Figure 7. A three-dimensional example illustrating the steps of the procedure described in the text.

It will be shown later that **M** is not an integer. Therefore the final arrangement of the unit cubes is shown as the last figure in Figure 7.

It is to be noted that while the drawing illustrating the geometric operation (Fig. 7) is entirely correct, the remaining outer surface as calculated by Expression **(14)** will be somewhat different than the actual remaining outer surface. This is because Expression **(14)** indicates the limit of possible minimum of remaining outer faces. The actual number of the remaining outer faces may be greater.

Readers: Do not follow your natural instincts. The entire process is based on accounting of the outer faces *per unit cube* no matter where a unit cube is positioned. The process is *not based* on accounting of the apparent outer surface as illustrated in Fig. 7a, below.

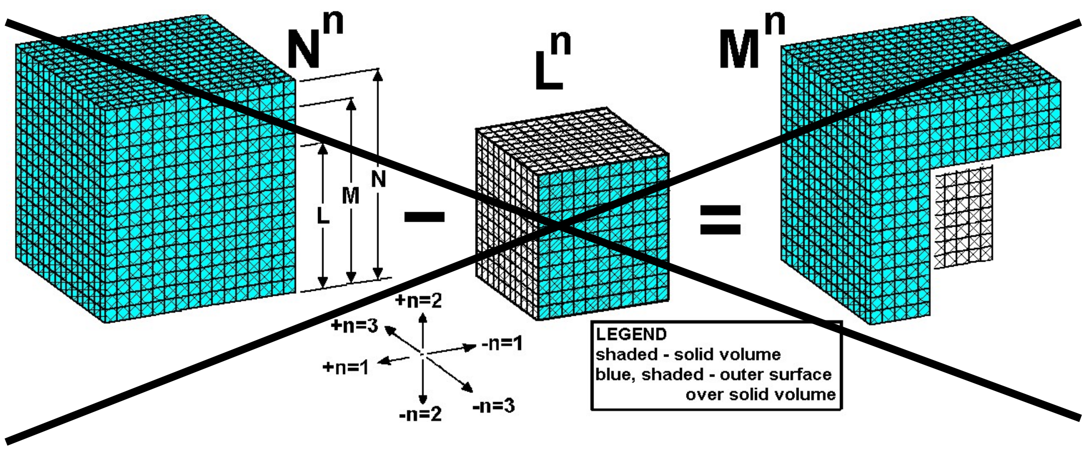

Figure 7a. This figure illustrates what must not be done.

9. PROVING THAT THERE ARE MORE OUTER FACES THAN NEEDED.

It was established what is the minimum possible outer surface area remaining after the subtraction of $\mathbf{L^n}$ from $\mathbf{N^n}$. Now it is to be demonstrated that this minimum possible outer surface area remaining after the subtraction of $\mathbf{L^n}$ from $\mathbf{N^n}$ is greater than the surface area required for $\mathbf{M^n}$, had **M** been an integer. In other words to show that there are too many outer faces (subscripts) for the units of the $\mathbf{M^n}$-super-cube to form a perfect super-cube.
That is, we need to prove that for $\mathbf{n > 2}$

$$\mathbf{2\ n\ N^{n-1} - L^n\ n/M \ > \ 2\ n\ \ M^{n-1}} \qquad \mathbf{(15)}$$

It will now have to be determined which side of **(15)** is greater than the other, and under what conditions. If both sides of **(15)** happen to be equal then the necessary condition for **M** to be an integer will be satisfied.
First, **L** and **M** will be expressed in terms of **N**. According to **(3)**, **L** is smaller than **N**. **L** and **N** can be interrelated by writing:

$$\mathbf{L^n \ = \ A\ N^n} \qquad \mathbf{(16)}$$

where

$$1.0 > A > 0.0 \qquad \mathbf{(17)}$$

Utilizing **(4)** and **(16)** **M** can also be expressed in terms of **N**. Thus:

$$M = (N^n - A N^n)^{1/n} \tag{18}$$

Utilizing expressions **(18)** and **(20)** in order to write **L** and **M** in terms of **N**, the LHS of **(15)** can be written in the following form:

$$2 n A N^n - \frac{n A N^n}{(N^n - A N^n)^{1/n}} \tag{19}$$

Utilizing **(18)** the RHS of **(15)** can be written in the following form:

$$2n(N^n - A N^n)^{(n-1)/n} \tag{20}$$

Utilizing **(19)** and **(20)** inequality **(15)** can be expressed in terms of **A** and **N** only:

$$2 n A N^n - \frac{n A N^n}{(N^n - A N^n)^{1/n}} \overset{?}{=} 2n(N^n - A N^n)^{(n-1)/n} \tag{21}$$

outer faces remaining after L^n was subtracted from N^n vs. outer faces corresponding to an array representing M^n

Divide both sides of **(21)** by $2n(N^n - A N^n)^{(n-1)/n}$ The result is:

$$\frac{N^{n-1}}{(N^n - A N^n)^{(n-1)/n}} - \frac{A N^n}{2 (N^n - A N^n)^{1/n}} \overset{?}{=} 1 \tag{22}$$

After simplifications **(22)** will become:

$$\frac{1}{(1 - A)^{(n-1)/n}} - \frac{A}{2 (1 - A)} \overset{?}{=} 1 \tag{23}$$

Multiply both sides of **(23)** by **2 (1 - A)**

$$\frac{2}{(1 - A)^{(n-2)/n}} - A \overset{?}{=} 2 (1 - A) \tag{23a}$$

It is apparent upon examination that both sides of **(23a)** are greater than zero. Both sides of **(23a)** will be raised to power **n**. After simplifications **(23a)** becomes:

$$(1 - A) \overset{?}{=} (1 - 0.5 A)^n \tag{23b}$$

Assume that **$0.5 \geq A > 0.0$** and consider cases when **$n = 1$**, **$n = 2$** and **$n \geq 3$**.

A **$0.5 \geq A > 0.0, \; n = 1$**

$$(1 - A) = (1 - 0.5 A) \qquad \textbf{(24)}$$

It is apparent that the RHS of **(24)** is always greater than the LHS of **(24)** That is, for **$n = 1$**, the minimum remaining quantity of the outer faces is smaller then the quantity of the outer faces of **M^n** if **M^n** had been an integer.

B **$0.5 \geq A > 0.0, \; n = 2$**

$$(1 - A) \overset{?}{=} (1 - 0.5 A)^2 \qquad \textbf{(25)}$$

After simplifications **(25)** becomes:

$$0 \overset{?}{=} A^2 \qquad \textbf{(25a)}$$

The left hand side of **(25a)** is always smaller than the right hand side. Consequently, for **$n = 2$**, the left hand side of **(25a)**, representing the minimum remaining quantity of the outer faces, is always smaller than the right hand side of **(25a)**, which represents the quantity of the outer faces of M^n, had **M^n** been an integer.

C. **$0.5 \geq A > 0.0, \quad n \geq 3$**

The components on both sides of **(23b)** will be compared, That is:

$$(1 - A) \overset{?}{=} (1 - 0.5 A)^n \qquad \textbf{(23b)}$$

Expression **(27b)** is a function of both **A** and **n**. To determine which side of **(23b)** is greater than, or equal to the other, and under what conditions, the functions representing both sides of **(23b)** will be investigated in the interval of

$0.5 \geq A > 0.0$.

First, compare both sides of **(27b)** when **$A \to 0.0$** and **$n \geq 3$**:

$$\lim_{\substack{A \to 0.0 \\ n \geq 3}} \text{LHS of (23b)} = 1 - 0.0 = 1.0 \qquad \textbf{(26a)}$$

$$\lim_{\substack{A \to 0.0 \\ n \geq 3}} \text{RHS of (23b)} = (1 - 0.0)^n = 1.0 \qquad \textbf{(26b)}$$

Next, compare both sides of **(23b)** for **A** approaching **0.5**. That is:

$$\mathbf{A \to 0.5 \ \ and \ \ n \geq 3}$$

Starting with $\mathbf{n = 3}$ both sides of **(27b)** are compared:

$$\lim_{\substack{A \to 0.5 \\ n = 3}} \text{LHS of (23b)} = \lim_{\substack{A \to 0.5 \\ n = 3}} (1 - A) = (1 - 0.5) = 0.5 \qquad \textbf{(27a)}$$

$$\lim_{\substack{A \to 0.5 \\ n = 3}} \text{RHS of (23b)} = \lim_{\substack{A \to 0.5 \\ n = 3}} (1 - A/2)^3 = (1 - 0.5/2.0)^3 = 0.422 \qquad \textbf{(27b)}$$

Then, when $A \to 0.5$ and $n \to \infty$

$$\lim_{\substack{A \to 0.5 \\ n \to \infty}} \text{LHS of (23b)} = \lim_{\substack{A \to 0.5 \\ n \to \infty}} (1 - A) = 0.5 \qquad \textbf{(28a)}$$

and

$$\lim_{\substack{A \to 0.5 \\ n \to \infty}} \text{RHS of (23b)} = \lim_{\substack{A \to 0.5 \\ n \to \infty}} (1 - A/2)^n = 0.0 \qquad \textbf{(28b)}$$

The functions representing both sides of **(23b)** are illustrated in Figure 8. The left hand side of **(23b)** is represented by the top line. Its value varies depending on **A** between **1.0** at $\mathbf{A = 0.0}$ and **0.5** at $\mathbf{A = 0.5}$ for any value of **n**. The right hand side of **(23b)** is represented by the lower curve in Figure 8. Its value varies between **1.0** at $\mathbf{A = 0.0}$ and **0.422** at $\mathbf{A = 0.5}$. As indicated by **(28a)** and **(28b)** when $\mathbf{n \to \infty}$, the line representing the left hand side of **(23b)** remains unchanged, while the curve representing the right hand side of (23b) is lowered.

It is apparent from **(27a)**, **(27b)**, **(28a)** and **(28b)** that for $\mathbf{A = 0.5}$ the right hand side of **(27b)** is always greater than the left hand side of **(23b)** for $\mathbf{n \geq 3}$. To establish which side of **(23b)** is greater over the entire range of $\mathbf{0.5 \geq A > 0.0}$, the slopes of the functions representing both sides of **(23b)** will be evaluated. To find the slopes differentiate both sides of **(23b)** with respect to **A**:

$$\frac{d}{dA}[\text{LHS of (27b)}] = \frac{d}{dA}(1 - A) = -1.0 \qquad \textbf{(29a)}$$

and

$$\frac{d}{dA}[\text{RHS of (27b)}] = \frac{d}{dA}(1 - A/2)^n$$

$$= -0.5\, n\, (1-A)^{n-1} \qquad \textbf{(29b)}$$

It is apparent from the examination of **(29a)** that for $\mathbf{n > 2}$ the slope of the function representing the right hand side of **(23b)** is always steeper than the slope of the function representing the left hand side of **(23b)** for values of **A** close to **0.0**. When the value of **A** increases from **0.0** to **0.5**, the slope of the function representing the right hand side of **(23b)** decreases in its magnitude. It can be concluded that for $\mathbf{0.5 \geq A > 0.0}$ and $\mathbf{n > 2}$ the left hand side of **(23b)** is greater than the right hand side of **(23b)**.

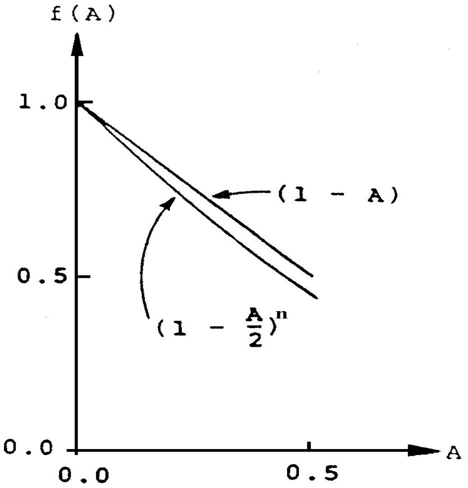

Figure 8.

Consequently, this proves that the left hand side of **(23)** is greater than the right hand side of **(23)** for $\mathbf{0.5 \geq A > 0.0}$ and $\mathbf{n \geq 3}$. This conclusion can be extended to $\mathbf{1.0 > A > 0.5}$, because, in effect, $\mathbf{L^n}$ would have taken place of $\mathbf{M^n}$ in equation **(4)**, and M^n would have taken place of $\mathbf{L^n}$.

Conclusion

It was shown above, that for $\mathbf{n > 2}$

$$2\,n\,N^{n-1} - L^n\,n\,/\,M > 2\,n\,M^{n-1} \qquad \mathbf{(30)}$$

That is, the minimum remaining quantity of the outer faces (the LHS of **(30)**) is greater than the quantity of the outer faces of M^n (the RHS of **(30)**), had $\mathbf{M^n}$ been an integer.

Thus, according to condition **(12)** since the number of the remaining outer faces is not equal to $\mathbf{2nM^{n-1}}$, **M** cannot be an integer. Consequently, for $\mathbf{n \geq 3}$, equation **(4)** cannot be satisfied.

The conclusion is different however, for **n** equal to **1** and n equal to **2**.

It was shown, for $\mathbf{n = 1}$ and $\mathbf{n = 2}$ that that the minimum remaining quantity of the outer faces is smaller than the quantity of the outer faces of $\mathbf{M^n}$, had $\mathbf{M^n}$ been an integer.

The *actual* remaining quantity of the outer faces (which can be greater than the minimum remaining quantity) may perhaps be equal to the quantity of the outer faces of $\mathbf{M^n}$, had $\mathbf{M^n}$ been an integer. In that case there is no surplus of the outer subscripts in an array representing M^n.

NOTE: It is to be noted that the inequalities in the text, as well as other expressions, are valid for continuous functions and not necessarily integers.

NUMERICAL EXAMPLE

This is an example in three dimensions with arbitrarily chosen numbers for **N** and L. The graphical illustration is presented in Figure A1.

Assume: $L < M < N$

L, N and n are integers, M = ?

L^n, M^n, N^n are all integers

Choose: **n = 3, L = 5, N = 8**

Therefore: $\mathbf{L^3 = 125}$, $\mathbf{N^3 = 512}$ and from equation (4) → $\mathbf{M^3 = 387}$

	$2\,n\,J^{n-1}$ Outer faces (10b)	$2\,n\,J^n$ Total faces (10a)	$2\,n\,(J^n - J^{n-1})$ Inner faces (10c)
L	$2 * 3 * 5^{3-1} = 150$	$2 * 3 * 5^3 = 750$	$750 - 150 = 600$
M	$2 * 3 * 387^{2/3} = 318.6338$	$2 * 3 * 387 = 2322$	$2322 - 318.6338 = 2003.3662$
N	$2 * 3 * 8^{3-1} = 384$	$2 * 3 * 8^3 = 3072$	$3072 - 384 = 2688$

where **J** is an integer.

Outer faces of $\mathbf{M^n}$ allocated to the units of an array representing $\mathbf{L^n}$:

$$\frac{2\,nM^{n-1}}{M^n}\,L^n = \frac{2*3*[387^{1/3}]^{3-1}}{[387^{1/3}]^3}\,5^3 = 102.91791$$

In the geometrical interpretation the term $\mathbf{nL^n/M}$ represents the largest possible outer surface area by which the outer surface of a cube representing $\mathbf{N^n}$ could be reduced when $\mathbf{L^n}$ is subtracted from $\mathbf{N^n}$.

Therefore, it can be concluded from geometrical considerations that the quantity of the outer faces which can be subtracted from $\mathbf{N^n}$, while satisfying expression (10b), is:

$$n\,L^n \,/\, (M^n)^{1/n} = 3*5^3 \,/\, (387)^{1/3} = 51.45895$$

[The quantity of outer faces which remains after $\mathbf{L^3}$ is subtracted from $\mathbf{N^3}$] $= 2\,n\,N^{n-1} - n\,L^n / (M^3)^{1/3}$

$$= 2*3*8^{3-1} - 3*5^3 / 7.28736 = 384 - 51.45895$$

$$= 332.54$$

Outer faces of $\mathbf{M^3}$ calculated by expression (10b) $= 2\,n\,M^{n-1} = 318.6338$

Therefore:

[The quantity of outer faces which remain after $\mathbf{L^3}$ is subtracted from $\mathbf{N^3}$] > [Outer faces of $\mathbf{M^3}$ calculated by expression (10b)]

$$332.54 > 318.6338$$

That is:

$$2\,n\,N^{n-1} - n\,L^{n} / M > 2\,n\,M^{n-1}$$

Conclusion: The quantity of the outer faces remaining after $\mathbf{L^3}$ was subtracted from $\mathbf{N^3}$ is greater than the quantity of the outer faces calculated by expression (10b).

Thus:

$$Q[M^{n}] \neq 2\,n\,M^{n-1}$$

where **Q**[ ] denotes the outer surface of a cube representing a particular array.

It is required that

$$Q[M^{n}] = 2\,n\,M^{n-1}$$

for **M** to be an integer. It was shown that

$$Q[M^{n}] > 2\,n\,M^{n-1}$$

Consequently, the necessary condition for **M** to be an integer was not satisfied and **M** is not an integer.

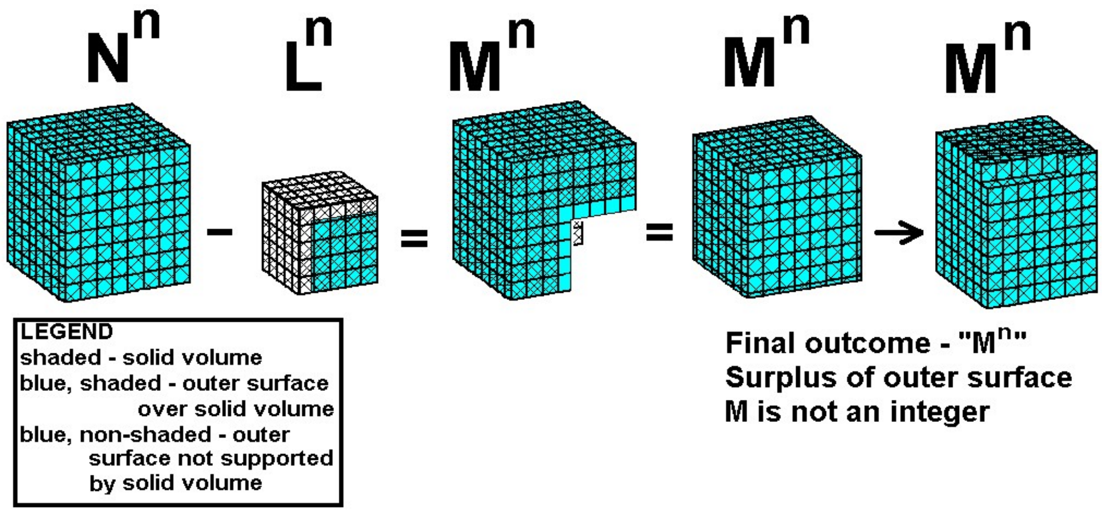

Figure A1. A graphical illustration of the numerical example