Verification and Validation of CFD Simulations

F. Stern[1], R. Wilson[1], H. Coleman[2], and E. Paterson[1]

1. Iowa Institute of Hydraulic Research, University of Iowa; 2. Propulsion Research Center, University of Alabama in Huntsville

Tutorial

2001 Fluids Engineering Division Summer Meeting

May 29 - June 1, 2001, New Orleans, Louisiana

Sheraton New Orleans Hotel

Outline

- Background
- Approach
- Overall Verification and Validation Methodology
- Verification Procedures
- Validation Procedures
- Example for RANS CFD Code
- Conclusions

Background

- Discussion and methodology for estimating errors and uncertainties in CFD simulations has reached a certain level of maturity
  - Editorial policies and early recognition of importance and need for "quality control" (ASME JFE, 1993, Vol. 115, pp. 339-340)
  - Increased attention and recent progress on common terminology through published guidelines (AIAA, 1998)
  - Advocacy and detailed methodology in recent textbook (Roache, 1998)
  - Numerous case studies, including a special AIAA J. issue (e.g., Mehta, 1998)

Background

- Progress accelerated in response to the urgent need for achieving consensus on concepts and definitions and useful methodology and procedures
  – CFD is applied to increasingly complex geometry and physics and integrated into the engineering design process
  – Realization simulation-based design
  – Other uses of CFD such as simulating flows for which experiments are difficult
- In spite of progress and urgency, various viewpoints have not converged and current approaches fall short of providing practical methodology and procedures

Approach

- Comprehensive, pragmatic approach to verification and validation (V&V) methodology and procedures for estimating errors and uncertainties for industrial CFD
- Already developed CFD code without requiring source code
- Specified objectives, geometry, conditions, and available benchmark experimental data
- Developed for RANS CFD codes, but should also be applicable to boundary-element methods and some aspects LES and DNS

Approach

- Definitions of errors and uncertainties consistent with experimental uncertainty analysis
- Concepts, definitions, and equations derived for simulation errors and uncertainties provide overall mathematical framework
- Numerical error treated both as deterministic or stochastic
- Generalized Richardson extrapolation for J input parameters and correction factors
- Use of quantitative estimates for errors and uncertainties constitute new V&V approach

Approach

- Previous work on verification (Stern et al., 1996) extended and and combined with subsequent work on validation (Coleman and Stern, 1997)
  - Stern, F., Wilson, R.V., Coleman, H., and Paterson, E., "Verification and Validation of CFD Simulations," Iowa Institute of Hydraulic Research, The University of Iowa, IIHR Report No. 407, September 1999 (in review ASME JFE)
- Nearly two years experience through International Towing Tank Conference (ITTC) community and Gothenburg 2000 Workshop on CFD in Ship Hydrodynamics (Larsson et al., 2000)

Overall V&V Methodology
Concepts and Definitions

- ***Error***, δ - difference between a simulation value (or an experimental value) and the truth
- ***Error estimate***, δ^* - estimate of both *sign and magnitude* of error δ
- ***Uncertainty estimate***, U - estimate of *magnitude* (but not sign) of error such that interval $\pm U$ contains true value 95 times out of 100

Overall V&V Methodology

Concepts and Definitions

- ***Simulation error***, δ_S - difference between simulation S and truth T postulated as comprised of addition of modeling and numerical errors

$$\delta_S = S - T = \delta_{SM} + \delta_{SN}$$

- The uncertainty equation corresponding to the error equation is

$$U_S^2 = U_{SM}^2 + U_{SN}^2$$

Overall V&V Methodology

Concepts and Definitions

- For certain conditions, the numerical error δ_{SN} can be considered as

$$\delta_{SN} = \delta^*_{SN} + \varepsilon_{SN}$$

  - where δ^*_{SN} is an estimate of sign and magnitude of δ_{SN} and ε_{SN} is the error in that estimate

- The corrected simulation value S_C (numerical benchmark) is

$$S_C = S - \delta^*_{SN} = T + \delta_{SM} + \varepsilon_{SN}$$

- With error and uncertainty equations

$$\delta_{S_C} = S_C - T = \delta_{SM} + \varepsilon_{SN} \qquad U^2_{S_C} = U^2_{SM} + U^2_{S_C N}$$

Overall V&V Methodology

Concepts and Definitions

- *Verification* - assessment of numerical uncertainty U_{SN} and when conditions permit, estimating the sign and magnitude of the numerical error δ^*_{SN} itself and the uncertainty $U_{S_C N}$ in that estimate

- *Validation* - assessment of modeling uncertainty U_{SM} by using benchmark data and, when conditions permit, estimating the sign and magnitude of the modeling error δ_{SM} itself

Overall V&V Methodology

Verification

- Numerical errors decomposed into contributions from iteration number grid size, time step, and other parameters

$$\delta_{SN} = \delta_I + \delta_G + \delta_T + \delta_P = \delta_I + \sum_{j=1}^{J} \delta_j$$

$$U_{SN}^2 = U_I^2 + U_G^2 + U_T^2 + U_P^2 = U_I^2 + \sum_{j=1}^{J} U_j^2$$

- Similarly,

$$\delta_{SN}^* = \delta_I^* + \sum_{j=1}^{J} \delta_j^*$$

$$S_C = S - (\delta_I^* + \sum_{j=1}^{J} \delta_j^*) = T + \delta_{SM} + \varepsilon_{SN}$$

$$U_{S_C N}^2 = U_{I_C}^2 + \sum_{j=1}^{J} U_{j_C}^2$$

Overall V&V Methodology

Validation

- The comparison error E is defined by

$$E = D - S = \delta_D - \delta_S = \delta_D - \left(\delta_{SMA} + \delta_{SPD} + \delta_{SN}\right)$$

  - with δ_{SM} decomposed into errors from modeling assumptions δ_{SMA} and use of previous data δ_{SPD}

- The uncertainty U_E in the comparison error is

$$U_E^2 = U_D^2 + U_{SMA}^2 + U_{SPD}^2 + U_{SN}^2$$

- Standard methodology and procedures available for estimating U_D (Coleman and Steele, 1999)

$^{2}_{STE} = U^2_{SPD} + U^2_{SN}$

Overall V&V Methodology

Validation

- Ideally, like to postulate if $|E|<U_{E,}$ validation achieved; however, no known method for estimating U_{SMA}
- More stringent alternative is, $|E|<U_{V,}$ where U_V is validation uncertainty

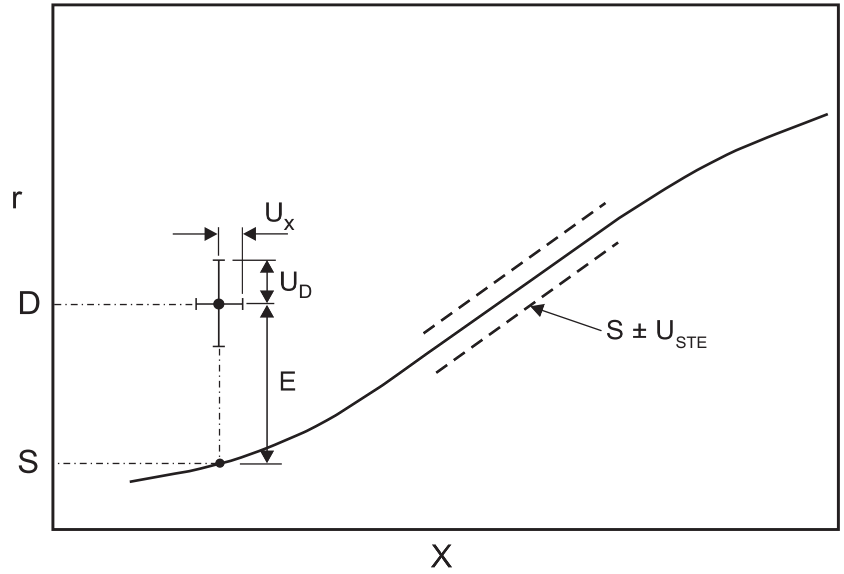
$$U_V^2 = U_E^2 - U_{SMA}^2 = U_D^2 + U_{SPD}^2 + U_{SN}^2 = U_D^2 + U_{STE}^2$$

  – U_{STE} is the total estimated simulation uncertainty

- If $|E|<U_V$ then validation achieved at U_V level
- If $|E|>>U_V$ then $E \approx \delta_{SMA}$

Overall V&V Methodology

Validation



Definition of comparison error

Overall V&V Methodology

Validation

- Corrected comparison error

$$E_C = D - S_C = \delta_D - \left(\delta_{SMA} + \delta_{SPD} + \varepsilon_{SN}\right)$$

- Corrected validation uncertainty

$$U_{V_C}^2 = U_{E_C}^2 - U_{SMA}^2 = U_D^2 + U_{SPD}^2 + U_{S_C N}^2$$

  - S_C and E_C can either be larger or smaller than S and E
  - U_{E_C} and U_{V_C} should be smaller than U_E and U_V

Verification Procedures

Convergence Studies

- Verification procedures based on

$$S = S_C + (\delta_I^* + \sum_{j=1}^{J} \delta_j^*)$$

- Parameter convergence studies
  - use multiple (m) solutions by varying the k^{th} input parameter Δx_k while holding all other parameters constant
  - input parameters (step sizes) defined such that $\Delta x_k \to 0$ for finest resolution
  - uniform parameter refinement $r_k = \Delta x_{k_2} / \Delta x_{k_1} = \Delta x_{k_3} / \Delta x_{k_2}$
    - Not required
- Solutions corrected for iterative errors

$$\hat{S}_{k_m} = S_{k_m} - \delta_{I_{k_m}}^* = S_C + \delta_{k_m}^* + \sum_{j=1, j \neq k}^{J} \delta_{j_m}^*$$

Verification Procedures

Convergence Studies

- Convergence studies require a minimum of *m=3* solutions $\left(\hat{S}_{k_1}, \hat{S}_{k_2}, \hat{S}_{k_3}\right)$ corresponding to fine, medium, and coarse values for the k^{th} input parameter
- Solution changes ε for medium-fine and coarse-medium solutions and their ratio R_k are defined by

$$\varepsilon_{k_{32}} = \hat{S}_{k_3} - \hat{S}_{k_2}$$

$$\varepsilon_{k_{21}} = \hat{S}_{k_2} - \hat{S}_{k_1}$$

$$R_k = \varepsilon_{k_{21}} \Big/ \varepsilon_{k_{32}}$$

Verification Procedures

Convergence Studies

- *Converging condition*: $0 < R_k < 1$, monotonic convergence and generalized RE is used to estimate U_k or δ_k^* and U_{k_C}
- *Oscillatory condition*: $R_k < 0$, solutions exhibit oscillations, which may be erroneously identified as convergent or divergent
  - uncertainties estimated using upper (S_U) and lower (S_L) bounds of solution oscillation and require more than 3 solutions

$$U_k = \frac{1}{2}(S_U - S_L)$$

- *Diverging condition*: $R_k > 1$, solutions exhibit divergence and errors and uncertainties can not be estimated

Verification Procedures

Monotonic Convergence: Generalized RE

- Generalized Richardson Extrapolation (RE) for convergent condition
  - Modified and numerical error equations show error terms $\delta^*_{k_m}$ are in the form of a power series expansion in input parameter

$$\hat{S}_{k_m} = S_C + \sum_{i=1}^{n} (\Delta x_{k_m})^{p_k^{(i)}} g_k^{(i)} + \sum_{j=1, j \neq k}^{J} \delta^*_{j_m}$$

  - Subtraction of multiple solutions eliminates $\delta^*_{j_m}$ terms and provides equations for S_C, $p^{(i)}{}_k$, and $g^{(i)}{}_k$ (assuming $p^{(i)}{}_k$, and $g^{(i)}{}_k$ independent Δx_k)
  - Since each term contains 2 unknowns, $m=2n+1$ solutions required (i.e., for $n=1$, $m=3$ and for $n=2$, $m=5$, etc.)

Verification Procedures

Monotonic Convergence: Generalized RE

- For *m*=3 solutions, only leading term estimated

$$\delta^*_{k_1} = \delta^*_{RE_{k_1}} = \frac{\varepsilon_{k_{21}}}{r_k^{p_k} - 1} \qquad p_k = \frac{\ln\left(\varepsilon_{k_{32}} / \varepsilon_{k_{21}}\right)}{\ln\left(r_k\right)}$$

  – for sufficiently small Δx_k solutions in the asymptotic range such that higher-order terms negligible

  – achieving asymptotic range for practical geometries and conditions not possible and m>3 undesirable from resource point of view

Verification Procedures

Estimating Errors and Uncertainties using Generalized RE with Correction Factors

- Results for analytical benchmarks (1D wave and 2D Laplace equations) show that when solutions not in asymptotic range δ^*_{RE} correct form but p_k poorly estimated
- Analysis results suggests concept of correction factors C_k
  - provide quantitative metric to determine proximity of solutions to asymptotic range
  - account for effects of higher-order terms
  - use for defining and estimating errors and uncertainties

Verification Procedures

Estimating Errors and Uncertainties using Generalized RE with Correction Factors

- Multiplication δ^*_{RE} by C_k provides estimate $\delta^*_{k_1}$ accounting for effects higher order terms

$$\delta^*_{k_1} = C_k \delta^*_{RE_{k_1}} = C_k \left(\frac{\varepsilon_{k_{21}}}{r_k^{p_k} - 1} \right)$$

- C_k based on leading and first two terms

$$C_k^{(1)} = \frac{r_k^{p_k} - 1}{r_k^{p_{k_{est}}} - 1}$$

$$C_k^{(2)} = \frac{(\varepsilon_{23_k} / \varepsilon_{12_k} - r_k^{q_{k\,est}})(r_k^{p_k} - 1)}{\left(r_k^{p_{k\,est}} - r_k^{q_{k\,est}}\right)\left(r_k^{p_{k\,est}} - 1\right)} + \frac{(\varepsilon_{23_k} / \varepsilon_{12_k} - r_k^{p_{k\,est}})(r_k^{p_k} - 1)}{(r_k^{p_{k\,est}} - r_k^{q_{k\,est}})(r_k^{q_{k\,est}} - 1)}$$

  - where p_{kest}, q_{kest} are improved estimates of orders of accuracy (e.g., modified equation or simplified geometry with similar grid expansion)
  - C_k<1 or C_k>1 indicates that the leading-order term over predicts or under predicts the error, respectively

Verification Procedures

Estimating Errors and Uncertainties using Generalized RE with Correction Factors

- For C_k<1 or C_k>1 and lacking confidence, U_k is estimated, but not δ_k^* and U_{kc}

$$U_k = \left| C_k \delta^*_{RE_{k_1}} \right| + \left| (1 - C_k) \delta^*_{RE_{k_1}} \right|$$

- For $C_k \approx 1$ and having confidence, δ_k^* and U_{k_C} are estimated

$$\delta^*_{k_1} = C_k \delta^*_{RE_{k_1}} = C_k \left(\frac{\varepsilon_{k_{21}}}{r_k^{p_k} - 1} \right) = \frac{\varepsilon_{k_{21}}}{r_k^{p_{th}} - 1}$$

$$U_{k_C} = \left| (1 - C_k) \delta^*_{RE_{k_1}} \right|$$

  - In the limit of the asymptotic range, C_k=1, $\delta^*_{k_1} = \delta^*_{RE_{k_1}}$, and U_{k_C}=0

Verification Procedures

Estimating Errors and Uncertainties using Generalized RE with Factors of Safety

- Roache (1998) proposes factor of safety approach

$$U_k = F_S \left| \delta^*_{RE_{k1}} \right|$$

- Can be extended for estimate of corrected simulation numerical uncertainty

$$U_{k_C} = \left(F_S - 1 \right) \left| \delta^*_{RE_{k1}} \right|$$

- In this approach, fixed percentage of three-gird error estimate used to define uncertainty: F_S=1.25 for careful grid study otherwise =3
  - Results for analytical benchmark show F_S overly conservative compared to C_k approach

Verification Procedures

Fundamental and Practical Issues

- Fundamental Issues
  - Convergence power series expansion
  - Assumption $p^{(i)}_k$, and $g^{(i)}_k$ independent Δx_k
  - Estimating $p_{k_{est}}$ based on theoretical values or solutions for simplified geometry and conditions with stretched grids
  - C_k vs. F_S or other approaches
- Practical Issues
  - For complex flows with relatively coarse girds, solutions far from asymptotic range such that while some variables convergent other variables may be oscillatory or even divergent
  - p_k shows variability between different variables same grid study and same variables different grid studies
  - More than 3 grids required
  - Selection parameter refinement ratio
  - Nature asymptotic range for practical applications unknown
  - Interpretation results important since limited experience for guidance

Validation Procedures

- Six combinations of $|E|$, U_V, and U_{reqd} (program validation requirement)
- 1. $|E| < U_V < U_{reqd}$
- 2. $|E| < U_{reqd} < U_V$
- 3. $U_{reqd} < |E| < U_V$
- 4. $U_V < |E| < U_{reqd}$
- 5. $U_V < U_{reqd} < |E|$
- 6. $U_{reqd} < U_V < |E|$
- In cases 1, 2, and 3, validation achieved at U_V level
- In cases 4, 5, and 6, validation not achieved at U_V. If $E >> U_V$ then

$$E \approx \delta_{SMA}$$

- In cases 1 and 4, validation successful programmatically
- Similar conclusions for corrected simulation results

Example for RANS CFD Code

- CFDSHIP-IOWA
- Series 60 cargo/container ship
  - ITTC benchmark data (Toda et al., 1992)
  - Conditions
    - Froude number Fr = 0.316
    - Reynolds number $Re = 4.3\text{x}10^6$
  - V&V for resistance C_T (integral variable) and wave profile ζ (point variable)

Example for RANS CFD Code

Grid Studies

- *Grid refinement ratio* $r_G = \sqrt{2}$
- *m*=4 grids with systematic grid refinement in each coordinate direction enables two separate grid studies: grids 1-3 (GS1) and grids 2-4 (GS2)

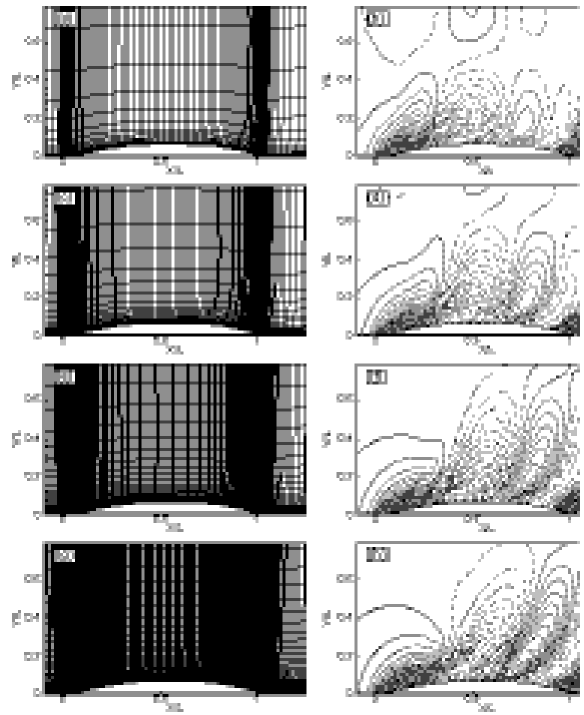
Table 1. Grid dimensions and *y*+ values for grid refinement studies.

Grid	*Grid Dimensions*	*Total Number of points*	*y+*
1	*287x78x43*	*876,211*	*0.7*
2	*201x51x31*	*317,781*	*1*
3	*144x36x22*	*114,048*	*1.4*
4	*101x26x16*	*42,016*	*2*

Example for RANS CFD Code

Grid Studies

- Grids and wave contours



Example for RANS CFD Code

Verification for Resistance

- Verification performed for iterative and grid convergence

$$U_{SN}^2 = U_I^2 + U_G^2$$

- Limiting order of accuracy estimated as formal order of accuracy of the CFD code

$$p_{k_{est}} = p_{k_{th}} = 2.0$$

- Iterative convergence negligible (i.e., at least one order of magnitude smaller grid convergence)

Grid 1 U_I=0.07%S_1

Grid 2 U_I=0.02%S_1

Grid 3 U_I=0.03%S_1

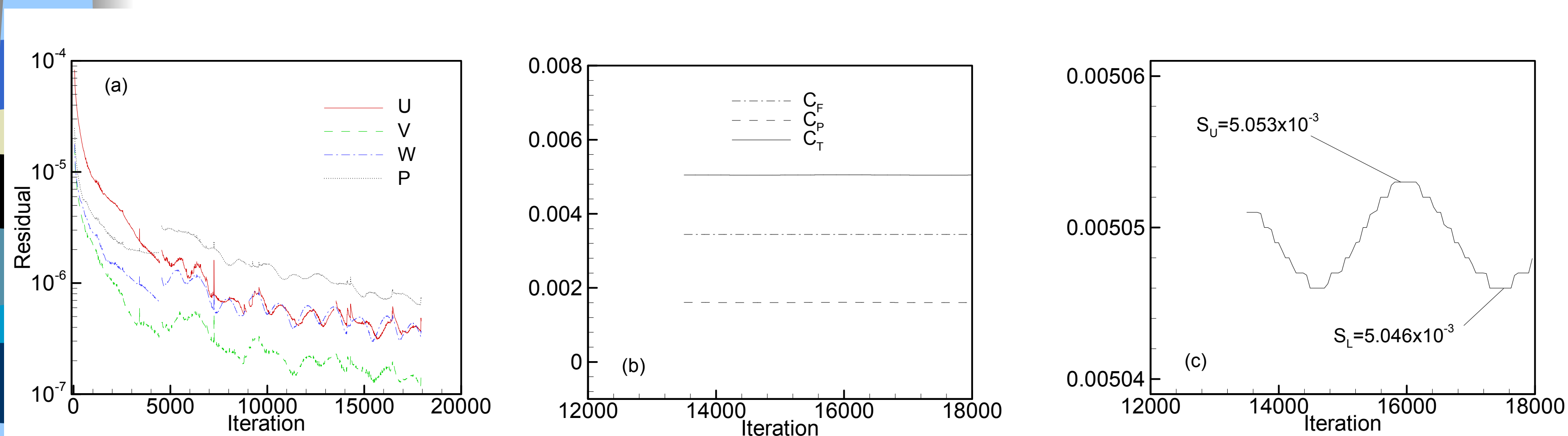
Grid 4 U_I=0.01%S_1

S_1=solution on finest grid

Example for RANS CFD Code

Verification for Resistance

- Iteration history: solution change; forces; and magnified C_T last two periods: fine grid; 4 order of magnitude drop in residuals; U_I=.07%S_1



Example for RANS CFD Code

Verification for Resistance

- Monotonic convergence; variability p_G and counter expectation

Table 2. Grid convergence study for total C_T, pressure C_P, and frictional C_F resistance ($x10^{-3}$) for Series 60.

Grid	*S_4 (grid 4)*	*S_3 (grid 3)*	*S_2 (grid 2)*	*S_1 (grid 1)*	*Data*
C_T ε	6.02	5.39 -10%	5.11 -5.2%	5.05 -1.2%	5.42
C_P ε	1.88	1.61 -14%	1.60 -0.6%	1.60 0.0%	$C_R = 2.00$
C_F ε	4.14	3.69 -11%	3.51 -4.9%	3.45 -1.7%	3.42 ITTC

% S_G.

Table 3. Verification of C_T ($x10^{-3}$) for Series 60.

Study	R_G	p_G	C_G
1 *(grids 1-3)*	0.21	4.4	3.7
2 *(grids 2-4)*	0.44	2.3	1.3

% S_G

Example for RANS CFD Code

Verification for Resistance

- C_F=70%C_T and within 1%C_F(ITTC)
- For GS1, C_P grid independent and C_F convergent with p_G>p_{th}
- For GS2, both convergent with p_G>p_{th}
- Fact that C_P and C_F converge with different p_G and depend different physics helps explain variability

Table 4 Verification of C_P and C_F ($x10^{-3}$) for Series 60.

Study	C_P			C_F		
	R_G	p_G	C_G	R_G	p_G	C_G
1 (grids 1-3)	0.00	-	-	0.33	3.2	2.0
2 (grids 2-4)	0.04	9.5	26	0.40	2.6	1.5

% S_1

Example for RANS CFD Code

Verification for Resistance

- Error and uncertainty values reasonable in consideration number grid points
- F_S approach less conservative, which is opposite results for analytical benchmark

Table 5. Errors and uncertainties for C_T ($x10^{-3}$) for Series 60.

Grid	C_T uncorrected		C_T corrected			
	U_G (C_G)	U_G (F_S)	δ_G^*	U_{G_C} (C_G)	U_{G_C} (F_S)	S_C
1	2.1%	0.5%	1.2%	0.9%	0.1%	4.99
2	6.7%	5.6%	5.5%	1.1%	1.1%	4.83

% S_1

Example for RANS CFD Code

Verification for Resistance

- Next finer $r_k = \sqrt{2}$ grid requires 2.4M grid points
- Expect $U_G \to U_{G_C}$ but with U_I similar order magnitude
- From resource point of view, accept present S_c and U_{G_c} for finest grid
- Based on overall verification: four solutions display monotonic convergence with $\delta_G^* > 0$ notwithstanding variability p_G which precludes complete confidence
- Additional solutions desirable for knowledge asymptotic range for practical applications

Example for RANS CFD Code

Validation for Resistance

- Uncorrected solution
  - For GS1, $|E| > U_V \Rightarrow E \approx \delta_{SMA} = 7\%D$ and U_D>U_{SN}
  - For GS2, $|E| < U_V$ so C_T validated at 7%D and U_{SN} >U_D
- Corrected Solution
  - For GS1 and GS2, $|E_c| >> U_{V_c} \Rightarrow E_c = \delta_{SMA} = 8\%D$ and $U_{S_cN} << U_D$

Table 6. Validation of uncorrected total resistance for Series 60.

Grid	*E%*	U_V%	U_D%	U_{SN}%
1	6.8	3.1	2.5	1.9
2	5.7	6.7	2.5	6.3

%D.

Table 7. Validation of corrected total resistance for Series 60.

Grid	E_C %	U_{V_C} %	U_D %	U_{S_CN} %
1	7.9	2.6	2.5	0.8
2	11	2.7	2.5	1.0

%D.

Example for RANS CFD Code

V&V for Wave Profile

- Wave height at free-surface hull intersection
- Convergence ratio and order of accuracy defined using L2 norms of $\varepsilon_{k_{21}}$ and $\varepsilon_{k_{32}}$

$$\langle R_G \rangle = \left\| \varepsilon_{G_{21}} \right\|_2 / \left\| \varepsilon_{G_{32}} \right\|_2$$

$$\langle p_G \rangle = \frac{\ln\left(\left\| \varepsilon_{G_{32}} \right\|_2 / \left\| \varepsilon_{G_{21}} \right\|_2 \right)}{\ln(r_G)}$$

- As with C_T, $U_I \ll U_G$ such that $U_{SN}=U_G$

Example for RANS CFD Code

Verification for Profile Average Wave Profile

- For both GS1 and GS2, monotonic convergence
- Uncertainties GS1=1/2GS2
- Trends p_G consistent expectation
- Uncertainty values reasonable in consideration number of grid points

Table 8. Profile-averaged verification results for wave profile for Series 60.

Study	R_G	p_G	C_G	U_G	U_{G_C}
1 (grids 1-3)	0.64	1.3	0.56	2.0%	0.9%
2 (grids 2-4)	0.68	1.1	0.47	4.1%	2.2%

$\%\zeta_{max}$.

Example for RANS CFD Code

Validation for Profile Average Wave Profile

- Uncorrected solution
  - For GS1, not validated at E=5.2%ζ_{max} but margin small
  - For GS2, nearly validated at 5.6%ζ_{max}
- Corrected Solution
  - Not validated but margins relatively small and $U_{S_cN} << U_D$

Table 9. Profile-averaged validation results for uncorrected wave profile for Series 60.

Grid	*E%*	U_V%	U_D%	U_{SN}%
1	5.2	4.2	3.7	2.0
2	5.6	5.5	3.7	4.1

%ζ_{max} .

Table 10. Profile-averaged validation results for corrected wave profile for Series 60.

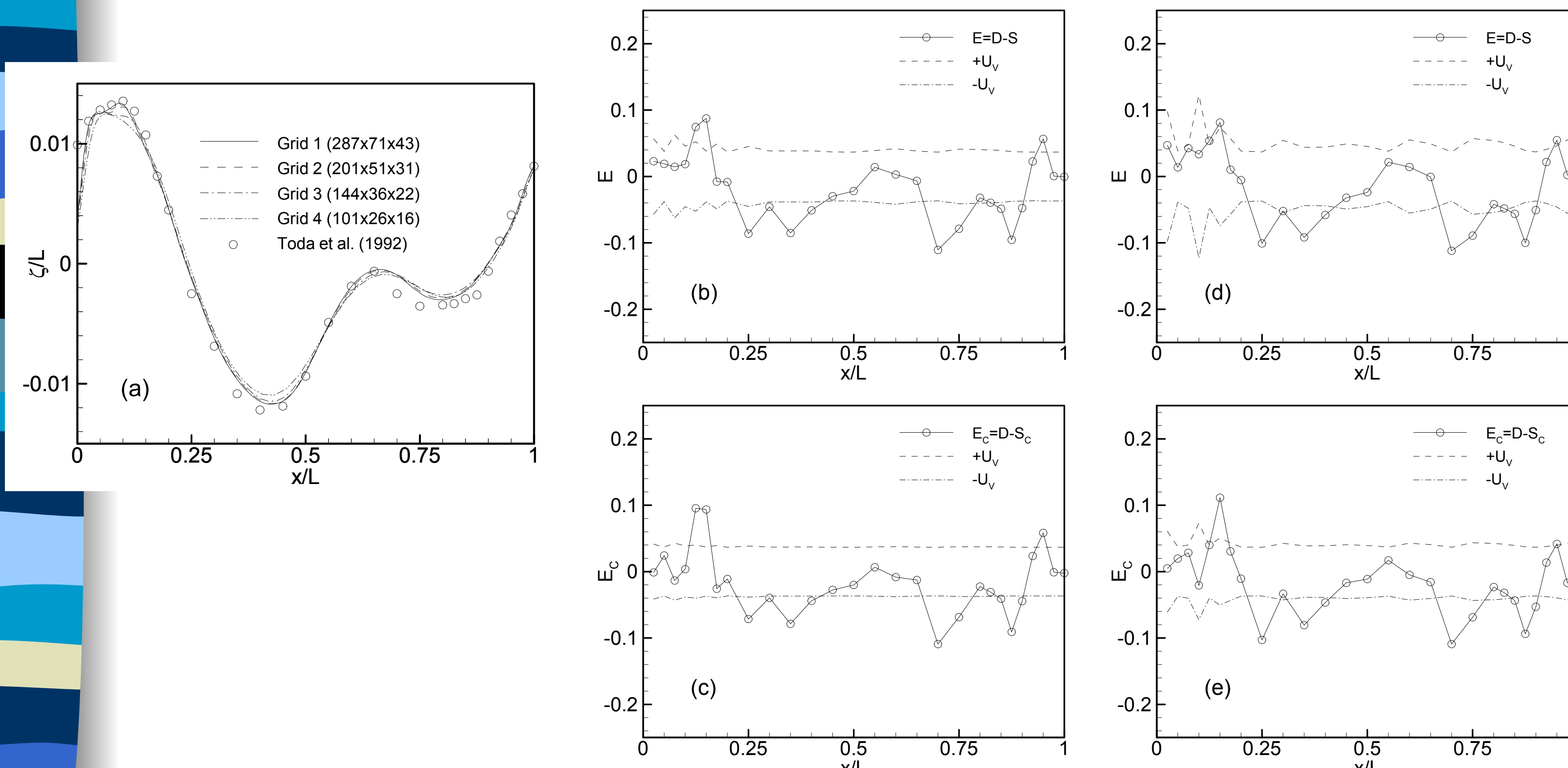
Grid	E_C %	U_{V_C} %	U_D %	U_{S_CN} %
1	5.6	3.8	3.7	0.9
2	6.6	4.3	3.7	2.2

%ζ_{max} .

Example for RANS CFD Code

V&V for Wave Profile

- Regions not validated indicate under prediction crests and troughs



Example for RANS CFD Code

Overall V&V Conclusions for C_T and Wave Profile

- C_T and ζ not validated due to 8%D and 6%ζ_{max} modeling errors
- Improve modeling assumptions for dynamic sinkage and trim, free surface boundary conditions, and turbulence for validation at 3%D and 4%ζ_{max} levels
- Reduction level validation U_V requires reduction U_D

Conclusions

- V&V methodology and procedures successful in assessing levels of verification and validation or modeling errors.
- For practical applications many issues
  - Solutions far from the asymptotic range
  - Analysis and interpretation results important in assessing variability for order of accuracy, levels of verification, and strategies for reducing numerical and modeling errors and uncertainties

Conclusions

- Future work on verification should focus on both fundamental and practical issues, as previously discussed
- V&V methodology and procedures should facilitate
  - Documented V&V studies for transition CFD codes to design
  - Sufficient number of documented solutions should enable accreditation of CFD code for a certain range of applications

Analytical Benchmarks

- 1D Wave equation

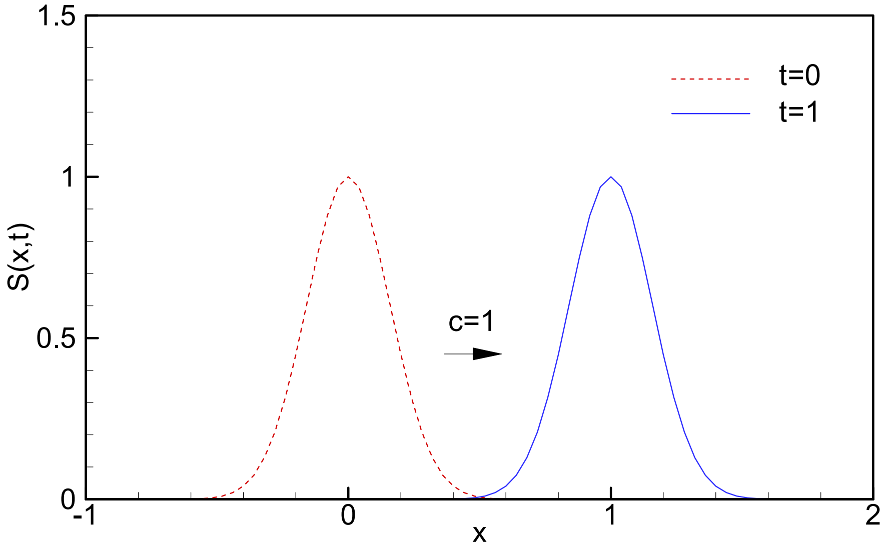
$$L_T(T) = L_M(M) = L_A(A) = \frac{\partial A}{\partial t} + c\frac{\partial A}{\partial x} = 0$$

$$IC: A(x,0) = A_0 \exp\left[-\frac{(x)^2}{B}\right]$$

$$BC: A(-\infty, t) = 0$$

- Exact analytical solution

$$A(x,t) = A_0 \exp\left[-\frac{(x-ct)^2}{B}\right]$$



Analytical Benchmarks

- Simulation error and uncertainty

$$\delta_S = S - A = \delta_{SN} \qquad U_S^2 = U_{SN}^2$$

- Corrected Simulation error and uncertainty

$$\delta_{S_C} = S_C - A = \varepsilon_{SN} \qquad U_{S_C}^2 = U_{S_C N}^2$$

- Verification

$$|E| = |A - S| < U_{SN} \qquad |E_C| = |A - S_C| < U_{S_C N}$$

Analytical Benchmarks

Verification results 1st order solution 1D wave equation